Distracted Learning[*]

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Abstract

We introduce a model of distracted learning mitigated by an optimal N-state memory buffer. A decision maker (Dug) continuously updates his beliefs given an Ito signal process. When the *termination shock* hits, he must choose a binary action, earning a higher payoff if his action matches the binary state. When a *distraction shock* occurs, he can only recall which one of N intervals *(memory states)* contained his belief when the distraction shock hit. He maximizes his expected payoff over interval partitions of $[0, 1]$, and his post-distraction beliefs. Dug is optimally indifferent between memory states at the threshold between memory states. His unoptimized value obeys smooth pasting; optimal values are "super-smooth pasted".

Dug is always harmed by distraction shocks, and gains at a termination shock if and only if he is sufficiently certain of the state. His initial value rises in the number of memory states and falls in the rate of termination shocks. Two memory states are sufficient to secure the full information value as termination shocks become vanishingly rare.

We consider two extensions: allowing the DM to choose the optimal number of memory slots (demand for memory) or the optimal precision of the Ito process (demand for information) given some increasing cost function.

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1 Introduction

Distraction has long been in the news, with agreement that smart phones and social networking have adversely impacted everything from educational outcomes[1] to highway death rates[2] (up 25% since 2010). Many studies have found that distraction results in incomplete learning, poor recollection, and reduced cognitive control.[3] This paper introduces a simple novel model of distraction that treats it as a constraint on rational Bayesian learning.

We merge finite state automata and standard Bayesian rational learning to isolate the pure impact of distraction, not conflating it with other cognitive limitations. Specifically, we assume that a fully rational decision maker, Dug,[4] optimally learns about a low or high *true state* (of the world) in continuous time, with the stochastic calculus model of Smith and Moscarini (2001). Dug knows that he faces unpredictable distractions that disrupt his train of thought. At those random moments, Dug forgets his precise posterior belief and reverts to a finite state belief buffer, as explored in Wilson (2014). Loosely, the memory states act as distraction insurance for the Bayesian rational Dug. What is the optimal design of this finite state automaton? How does it impact behavior, and how much does it impede his learning?[5]

Dug designs a memory policy at time zero, which — along with his current posterior *belief* (in the high true state) — dictates his belief after a distraction. Specifically, a memory policy partitions beliefs $[0, 1]$ into N *memory states.* Per usual, we assume these states are intervals. The memory policy assigns a belief q_n to each memory state n. Thereafter Dug continuously observes a flow signal of the state, obscured by Wiener noise. Between distraction shocks, his posterior belief in the high state adjusts continuously. Distraction shocks arrive exogenously at a fixed rate, according to a Poisson process, whereupon Dug forgets the past history of the signal process and distraction shocks, and only recalls his memory state n. When this arrival rate vanishes, our model reduces to fully rational Bayesian learning. Post-distraction, Dug resumes observing the signal process in continuous time, but now starting at the posterior q_n.

At some exponentially distributed time, Dug must choose a binary action, and earns an undiscounted terminal payoff that is positive if his action matches the state and zero if it does not. A prior bias weakly favors matching the high state. He chooses his memory policy at time zero to maximize his expected terminal payoff, perfectly forecasting the state conditional distributions of terminal beliefs induced by his chosen memory policy.

[1] See, for example, Digital Distractions In Class Linked to Lower Academic Performance (2023).
[2] See, for example, Wikipedia: motor vehicle fatality rate in US by year.
[3] See, for example, Ophir, Nass, and Wagner (2009).
[4] Inspired by Dug, the golden retriever from Pixar's Up (2009) with his squirrel distraction moments.
[5] Some recent studies have found that people can learn to mitigate distractions, even with uncertainty on when they will hit: Brain, Interrupted.

The design exercise is how to partition the belief space $[0,1]$ into memory states — a novel feature not yet explored in the finite memory literature — and what belief to endow Dug with in each memory state. Firstly, as in Wilson (2014), dynamically optimal beliefs should be Bayesian consistent (Lemma 1) in the following sense. Loosely, if Dug awakes in a memory state, aware that he has potentially been transitioning among memory states for an arbitrarily long time span, he should arrive at the Bayesian posterior belief q_n for that memory state. To characterize such beliefs, we solve for the state conditional Markov transition matrix between memory states at consecutive distraction shocks. Given this transition matrix, we solve for the relative frequency with which distraction shocks hit in each memory state. Applying Bayes' rule to these relative frequencies yields a necessary condition for Bayesian consistent beliefs. We show that these beliefs are unique for any interval partition (Proposition 2). Our proof is constructive, and gives an efficient algorithm for computing numerical solutions.

Optimality entails a novel second ingredient in our model. Fixing the post-distraction beliefs, rational Dug faces an optimal control exercise when choosing his interval policy in order to maximize his expected payoff. Dug has a Bellman value as a function of his current belief in each memory state. Within each memory state this value function is analogous to undistracted learning: It is continuously differentiable, and twice continuously differentiable except, perhaps, at the thresholds separating memory states. Also, it is locally strictly convex in his current belief inside each memory state: Incremental information has strictly positive value, even with a suboptimal interval policy. and contingent on a true state, value functions increase in Dug's confidence in that state. Proposition 1 derives an intuitive necessary condition for an optimal interval policy, assuming Bayesian consistency: At a memory state boundary, Dug is indifferent between recalling either memory state should distraction strike at that moment. Lemma 4 establishes that for any memory policy this indifference condition is equivalent to continuity of the second derivative of the unconditional value function at the boundary between memory regions. Intuitively, this *high order contact* or *super smooth pasting* condition ensures that the value of information that Dug extracts from the signal process is continuous. Notably, neither value matching nor smooth pasting are optimality conditions, as both are automatically met for any interval policy. Intuitively, Dug's future action is only impacted after the arrival of a distraction or decision shock, with no discontinuity induced by changing memory states before this.

The interplay of rational and distracted Dug is best seen in rational Dug's dynamic optimization. First, every distraction shock is formally a capital loss to him. Next, the decision shocks that terminate the learning model are capital *gains* if and only if he is sufficiently certain of the true state, since at these junctures he rationally fears that a distraction with destroy his near certainty, and replace it with the less certain insurance belief of the memory state. In other words, this is a unique learning model in which the Bayes rational value falls below the myopic payoff (Proposition 4).

Dug's initial value is strictly increasing in the number of memory states, and strictly decreasing in the rate of arrival of termination shocks, and the noise in the signal process (Proposition 5). We solve for the limits of his payoff in all exogenous parameters (Proposition 6). One surprising result emerges: Two memory states are sufficient for Dug to secure his full information value as the termination shock arrival rate converges to zero.

With two memory states, the optimal threshold between them is fully characterized by Bayesian consistent beliefs along with indifference at the boundary. This threshold rises as matching the high state becomes relatively less important, or as the high state becomes more likely; either way, the post-distraction beliefs in both memory states rise. As decision shocks become more frequent relative to distraction shocks, the optimal policy adjusts so that the post-distraction beliefs in the two memory states move closer together.

In an extension inspired by Smith and Moscarini (2001), we consider the demand for memory and the demand for information. For the former, we let Dug choose the number of memory states at time zero at some additive cost. We show that the demand for memory is robustly non-monotonic in the rate of distraction shocks. In all computed examples, the demand for memory is hump-shaped in the rate of distraction shocks for any convex cost of memory. For the demand for information, we instead allow Dug to choose the precision of the noisy signal at time zero. We find that the demand for precision is also robustly non-monotonic. In computed examples, Dug chooses more precision as distraction shocks become more common when precision is already high (i.e., when precision is cheap), but chooses worse information when distraction shocks become more common with costly precision.

Literature Review. There is a large literature on both finite automata and continuous Bayesian learning. We think ours is the first paper to merge the two. The most closely related paper is Wilson (2014), which also explores the optimal design of an automaton to process information about a binary state. But in that paper, both time and information were discrete: Each period, the DM observed one of finitely many signal realizations, updated her memory state accordingly, and retained no other information into the next period. In this paper, Dug continuously updates his beliefs until a distraction shock hits, at which point he loses all information beyond the current memory state. Both papers characterize the optimal use of finite memory states. But in the discrete setup, this required carefully randomized transitions and was tractable only when the chance of the information terminating was very small. In this paper, Dug more simply chooses the cutoff beliefs for transitions into each memory state. This provides enough structure to make the problem tractable for all parameters, and delivers comparative statics that were not possible in the discrete setup.

Both papers find that ex ante optimal memory policies are incentive compatible, according to the *modified multi-self consistency* notion introduced by Piccione and Rubinstein (1997). Namely, the problem can be reformulated as a multi-self game, with a new self controlling behavior each period in the discrete setting, and, in this paper, after each distraction shock. Under an ex ante optimal memory policy, no "self" can gain with a one-shot deviation, provided that they hold Bayesian (given available info) beliefs. Here, this means that at the cutoff belief to switch from memory state n to $n+1$, Dug is indifferent between the continuation payoffs he would obtain in these states (his "post-amnesia payoffs") *were a distraction shock to hit*. This is much stronger than continuity of his value function at the cutoffs (which also holds), and – since the post-amnesia payoffs for N states are straightforward to compute as the solution to N equations – this facilitates finding the optimal cutoffs .

Many papers have explored using finite-state automata to process information. In the pioneering contribution, Hellmann and Cover (1970) characterize ϵ-optimal automata for distinguishing between two hypotheses after an infinite sequence of signal observations, finding that the two memory states with the most extreme beliefs were optimally sticky, and that as the signal becomes perfectly informative, two memory states suffice to learn the truth. Chatterjee and Hu (2023) explore a similar set-up, focusing on approximately optimal heuristics based on ignoring noisy signal realizations and canceling out opposing ones. A related paper by Jehiel and Steiner (2020) study a memoryless agent (essentially a 1-state automaton) who can choose each period whether to acquire another signal or make a decision, and found that this generates biases such as confirmation bias and a salience effect.

There is also a growing literature on applications of bounded memory. Dow (1991) explores sequential search for the lowest price, assuming the DM can only recall how he categorized past prices (not their exact values), but could design the categories optimally. Lorrechio and Monte (2023b) studies information design with constrained capacity, where an infinite sequence of myopic agents choose whether to invest in a project that is either Good or Bad, their payoffs provide information to a designer, and the designer communicates one of finitely many ratings to future agents. Ratings are updated after each observation using a stochastic transition rule, and thus are analogous to memory states. In a follow-up paper, Lorrechio and Monte (2023a) consider an expert who fears developing a bad reputation (as in Ely and Valimaki (2003)) for always recommending high-cost actions, and who thus has incentives to recommend lower-cost actions than are actually optimal. They show that limiting information to a finite rating system can mitigate this problem. Relatedly, Ekmekci (2011) shows that restricting information to a finite grid can permit experts to permanently maintain "good" reputations (whereas Cripps, Mailath, and Samuelson (2004) showed good reputations are unsustainable in a world with unlimited information). Compte and Jehiel (2015) explored games played between players with finitely many mental states for tracking opponent behavior and carrying out decisions. Recently, some experimental papers have looked at the extent to which automaton models capture complexity constraints. Liu and Miao (2025) explores a sender-receiver game where a biased sender chooses when to stop the flow of truthful information. The receiver has a finite-state automaton to process information, and designs it strategically both to maximize his own payoff and to control the sender's information flow.

More broadly, we join a large literature on complexity costs and constraints. Much of the literature builds on the rational inattention model of Sims (2003), with more recent contributions in Matejka and McKay (2015) and Steiner, Stewart, and Matejka (2017). In these papers, the DM chooses both what to learn about, and how much to learn, given information costs proportional to its reduction in uncertainty. Fudenberg, Lanzani, and Strack (2017) study selective learning, where the DM recalls only a subset of experiences, but forms beliefs assuming that what he recalls is all that occurred.

There is a large literature on learning in continuous time. Karatzas (1984) solves for the optimal policy in a continuous time bandit model, where the state of each arm follows an Ito diffusion when actively engaged. Bolton and Harris (1999) consider a multi-agent generalization with positive learning externalities. We adopt the single agent, binary-state, binary-action, continuous time learning model from Smith and Moscarini (2001), but with a fixed experimentation level. Fudenberg, Strack, and Strzalecki (2018) consider a continuum state, binary action model with endogenous stopping with a constant flow cost of information.

2 The Model

We study how a rational risk-neutral decision maker (Dug) optimally designs a finite state automaton, to mitigate the information loss from random distractions that interrupt his continuous time learning.

Dug is uncertain of the true state of the world θ, fixed for all time at either H or L. His prior likelihood ratio favoring H is μ. Eventually, at Poisson rate $\delta > 0$, a *decision shock* terminates the problem. Dug must then immediately choose a one-time action, $\mathcal{H}$ or $\mathcal{L}$: He earns payoffs $\pi > 0$ if he matches state L, $\Pi > 0$ if he matches state H, and zero otherwise. Thus action $\mathcal{H}$ is optimal iff his terminal belief on state H exceeds $\hat{p} \equiv \pi/(\pi + \Pi)$. There is no discounting, and we assume WLOG that the prior bias favors action $\mathcal{H}$, i.e. $\mu\Pi \geq \pi$.

The model takes place in continuous time $t \in [0, \infty]$. Until the decision shock hits, Dug observes the scalar signal process $S(t)$, obscured by Wiener noise $W(t)$:

$$dS = s_\theta dt + \sigma dW \tag{1}$$

where $s_H = 1 = -s_L$, and $\sigma^{-1} > 0$ is the precision of the signal. As shown in Bolton and Harris (1999), while Dug observes (1), his unconditional beliefs $p(t)$ evolve according to:

$$dp = 2\sigma^{-1} p(1-p) dW \tag{2}$$

At Poisson rate $\alpha > 0$, Dug is hit by *distraction* (aka *amnesia*) shocks, and this erases all but a summary statistic that Dug designs as follows: He has $N > 0$ available *memory states*. Before learning begins, he partitions the belief space $[0, 1]$ into N distinct subsets, transitioning to memory state n whenever beliefs drift into the nth subset. He also designates an initial memory state n_0, and assigns a belief q_n, namely a posterior on the high state of the world, to each memory state n. We restrict to *interval policies* $I : [0, 1] \to \{1, 2, \ldots, N\}$, described by a sequence of thresholds $0 \equiv p_0 \leq p_1 \leq \cdots \leq p_{N-1} \leq p_N \equiv 1$.[6] Between distraction shocks, Dug is a rational Bayesian, and keeps track of both his exact belief $p(t)$ at every time t, and also the memory state that contains it, i.e. n with $p(t) \in [p_{n-1}, p_n]$. When a distraction shock occurs, he loses all information beyond the current memory state n and its associated belief q_n. Let $\mathcal{M}$ be the space of such memory policies $(\mathbf{q}, \mathbf{p})$ (where $\mathbf{q}$ is the vector of post-amnesia beliefs q_n and $\mathbf{p}$ the vector of interval policy cutoffs p_k).

As we show in the next section, any memory policy implies a unique cdf over terminal beliefs in each state, $G^\theta(\cdot|(\mathbf{q}, \mathbf{p}))$. Dug wishes to maximize the associated expected terminal payoff, namely $1/(1 + \mu)$ times the following *value*:

$$\pi G^L(\hat{p}|(\mathbf{q}, \mathbf{p})) + \mu\Pi(1 - G^H(\hat{p}|(\mathbf{q}, \mathbf{p}))) \quad \text{s.t. } p(0) = q_{n_0} \tag{3}$$

[6] Dow (1991) argued that interval policies are informationally optimal. Although this was not in a hidden state Bayesian world, interval policies are intuitively most informative in the sense of Blackwell (1953).

3 Preliminary Analysis and the Evolution of Beliefs

We now describe the evolution of beliefs: first continuously, between amnesia shocks, and then discretely as Dug transitions from one memory state to the next.

A. Forecasting Beliefs at the Next Shock. It is convenient to work with the log likelihood ratio $\ell(p) \equiv \log(p/(1-p))$. We show in Appendix A that it obeys the following linear homogeneous Ito process conditional on the state θ:

$$d^\theta \ell = 2s_\theta \sigma^{-2} dt + 2\sigma^{-1} dW \tag{4}$$

The cdf over future beliefs is well known for this case:[7] Given current value ℓ_0, the change in the log likelihood ratio $\ell - \ell_0$ at time t is normally distributed with mean $2s_\theta t\sigma^{-2}$ and variance $4t\sigma^{-2}$. Using this and the fact that the decision and amnesia shocks follow independent Poisson processes, we show in Appendix A that the probability that the change $\ell - \ell_0$ is at most x when the next amnesia shock hits is as follows in state H:

$$F^H(x) = \begin{cases} 1 - \frac{1+\xi}{2\xi} e^{\frac{x(1-\xi)}{2}} & \text{if } x > 0 \\ \frac{\xi-1}{2\xi} e^{\frac{x(\xi+1)}{2}} & \text{if } x < 0 \end{cases} \qquad \text{for } \xi \equiv \sqrt{1+2(\alpha+\delta)\sigma^2} \tag{5}$$

And by symmetry, the cdf over log likelihood ratio changes in state L is $F^L(x) = 1 - F^H(-x)$. Since amnesia and decision shocks are independent, F^H is also the cdf over $\ell - \ell_0$ at the next decision shock, conditional on it hitting before the next amnesia shock.

The amount of learning that transpires between shocks is fully determined by, and falling in, ξ, which is a monotone function of the scaled noise, i.e. the variance of the signal process multiplied by the arrival rate of shocks (termination plus distraction).

B. The Conditional Frequency Distribution over Memory States. We now derive the long-run distribution over memory states for memory policy $(\mathbf{q}, \mathbf{p})$. Let $\lambda^\theta_{n,k}(\mathbf{q}, \mathbf{p})$ be the chance of a transition from memory state n to memory state k between adjacent amnesia shocks, conditional on: (a) state θ, (b) no intervening decision shock, and (c) assuming post-amnesia belief q_n in state n. Between shocks, Dug is engaged in standard Bayesian rational learning, and so, noting that the cdf F^θ over beliefs at the next amnesia shock is independent

[7] See Example 2 on page 217 of Karlin and Taylor (1981).

of the memory policy, we have:

$$\lambda_{n,k}^{\theta}(\mathbf{q},\mathbf{p}) = F^{\theta}(\ell(p_k) - \ell(q_n)) - F^{\theta}(\ell(p_{k-1}) - \ell(q_n)) \tag{6}$$

Given these transition chances, the conditional distribution over memory states after t amnesia shocks, $\rho^{\theta,t} = (\rho_1^{\theta,t}, \dots, \rho_N^{\theta,t})$, obeys the difference equation (suppressing $\mathbf{p}$ and $\mathbf{q}$):

$$\rho^{\theta,t} = \lambda^{\theta}\rho^{\theta,t-1} \tag{7}$$

Now let $\eta \equiv \frac{\delta}{\alpha+\delta}$ denote the chance that a decision shock occurs before the next amnesia shock, and let Λ^{θ} be the matrix with (i,j)th entry $(1-\eta)\lambda_{j,i}^{\theta}$. With ρ^0 the initial distribution ($\rho_{n_0}^0 = 1$ in both states $\theta = H, L$), iterate (7) to obtain:[8]

$$\rho^{\theta} = \eta\rho^0 + \eta(1-\eta)\rho^{\theta,1} + \eta(1-\eta)^2\rho^{\theta,2} + \ldots = \eta(I - \Lambda^{\theta})^{-1}\rho^0 \tag{8}$$

The vector ρ^{θ} given by (8) is the frequency distribution for memory states across amnesia shocks given θ, and thus, ρ_n^{θ} is the chance that Dug will be in memory state n at the final amnesia shock prior to termination. We can use ρ^{θ} both to compute his payoff (from (3)), and also to determine the Bayesian beliefs $\mathbf{q}$ consistent with a given interval policy $\mathbf{p}$.

C. Bayesian Consistent Beliefs. A Bayesian observer who knows the memory policy, and that Dug just experienced a distraction shock in memory state n, would conclude that $\Pr[\theta = H] = \rho_n^H/(\rho_n^H + \rho_n^L)$. Thus, we call beliefs *Bayesian consistent* if they obey the following fixed point equation:

$$\frac{q_n}{1-q_n} = \mu\left(\frac{\rho_n^H}{\rho_n^L}\right) = \mu\frac{[(I - \Lambda^H(\mathbf{q},\mathbf{p}))^{-1}\rho^0]_n}{[(I - \Lambda^L(\mathbf{q},\mathbf{p}))^{-1}\rho^0]_n} \qquad \forall\ n \tag{9}$$

From Wilson (2014), this captures the fact that if Dug updates knowing he potentially has been wandering between memory states for an arbitrarily long period of time, he would arrive at the same posterior in memory state n as if he just accepts the belief q_n. But here there is a critical complication that did not arise in the discrete model. There, the DM transitioned based on signal observations, so transition chances were independent of his beliefs. Here, Dug instead transitions when his belief drifts into a new interval between shocks, and so the chance that this happens depends on his belief at the last shock. That is, Bayesian

[8] This part is identical to Wilson (2014), just counting amnesia shocks instead of discrete time periods, and with η the chance of termination before the next amnesia shock.

consistency demands that the beliefs $\mathbf{q}$ we use to compute transition chances and hence ρ^θ, agree with those we compute based on long-run frequencies ρ_n^H/ρ_n^L. We prove in Section 4 that equation (9) indeed has a unique solution $\mathbf{q}$ for any interval policy $\mathbf{p}$.

D. Reformulating Dug's Objective Function. Let $\mathcal{V}^\theta(\mathbf{q},\mathbf{p})$ denote the expected payoff starting at time zero, conditional on state θ. To compute $\mathcal{V}^\theta$, first let w_n^θ be the expected payoff starting just after an amnesia shock in memory state n given that the decision shock hits before another amnesia shock:[9]

$$w_n^H = \Pi\left(1 - F^H\left(\log\left(\frac{\pi(1-q_n)}{\Pi q_n}\right)\right)\right) \quad \text{and} \quad w_n^L = \pi F^L\left(\log\left(\frac{\pi(1-q_n)}{\Pi q_n}\right)\right) \tag{10}$$

With this, recalling that ρ_n^θ is the chance that Dug is in memory state n when hit by the final amnesia shock before termination, we can rewrite his objective function in (3) as:

$$\mu\mathcal{V}^H(\mathbf{q},\mathbf{p}) + \mathcal{V}^L(\mathbf{q},\mathbf{p}), \text{ where } \mathcal{V}^\theta(\mathbf{q},\mathbf{p}) = \sum_n \rho_n^\theta w_n^\theta \tag{11}$$

Finally, we calculate Dug's *post-amnesia payoffs*, which we will use to determine the optimal interval policy. Let ν_n^θ be the expected payoff immediately following an amnesia shock in memory state n, given θ, which are the unique fixed points to the contraction mapping:

$$T\nu_n^\theta = \eta w_n^\theta + (1-\eta)\sum_k \lambda_{n,k}^\theta \nu_k^\theta \tag{12}$$

In other words, starting the moment after a distraction shock in memory state n, Dug anticipates that either a decision shock comes next (chance η) and the value is w_n^θ, or another distraction shock hits next (chance $1-\eta$) and his rational learning takes him to the memory states with the computed chances in (6).

4 Optimal Memory Policies

We now characterize optimal memory policies. We first highlight that, unlike in the existing discrete-time automata models, Dug's objective function in (11) depends directly on his post-distraction beliefs $\mathbf{q}$, since these influence both the long-run distribution ρ^θ from (8) and the payoffs w_n^θ from (10). One could imagine that if Dug could freely choose the beliefs attached

[9]This is the payoff from matching the state θ, multiplied by the chance that between the final amnesia shock (in memory state n) and the decision shock, Dug's belief likelihood ratio drifts from $q_n/(1-q_n)$ above the threshold to choose action $\mathcal{H}$ when $\theta = H$, or below this threshold when $\theta = L$.

to his memory states, he might deviate from Bayesian consistency. But we find that this is in fact not optimal:

Lemma 1 *In an optimal memory, beliefs are Bayesian consistent, i.e. obey* (9)*.*

It is intuitive that optimal beliefs are Bayesian. After all, why would Dug want to endow his future self with beliefs that do not optimally condition on all available information?

The next result characterizes optimal interval thresholds $\mathbf{p}$, finding that Dug wants to transition to the memory state with the highest expected post-amnesia payoff:

Proposition 1 (FOCs for Interval Thresholds) *Fixing Bayesian consistent beliefs* $\mathbf{q}$*,*

$$p_n\nu_n^H + (1-p_n)\nu_n^L \lesseqgtr p_n\nu_{n+1}^H + (1-p_n)\nu_{n+1}^L \Rightarrow \frac{\partial\mathcal{V}(\mathbf{p},\mathbf{q})}{\partial p_n} \lesseqgtr 0$$

The optimal threshold p_n *leaves Dug indifferent between his expected post-distraction payoffs in memory states* n *and* $n+1$*:*

$$p_n\nu_n^H + (1-p_n)\nu_n^L = p_n\nu_{n+1}^H + (1-p_n)\nu_{n+1}^L \tag{13}$$

The memory states act as insurance for Dug. If a distraction shock were to hit when rational Dug is exactly indifferent between memory states, he would secure identical payoffs from exercising either memory state insurance. This relates to existing results on incentive compatibility of optimal policies.[10] Namely, viewing Dug as a new "self" after each amnesia shock, this says that he cannot improve his post-amnesia payoff with a one-shot deviation. The slight difference[11] is that Dug's post-distraction payoff differs from his continuation payoff at the time he transitions, since a distraction has not yet happened at this point. But since the signal process, learning between shocks, and the arrival rate of shocks are independent of the memory policy, the interval cutoffs only impact the mapping from current beliefs to post-amnesia beliefs.

The proof of this result (in Appendix B) shows more strongly that Dug could not achieve any higher value even if he could condition his interval policy on where amnesia last struck, thus potentially associating each memory state n with a different vector $\mathbf{p}$ of thresholds. All of Dug's post-amnesia selves would choose the same vector of thresholds.

[10]Piccione and Rubinstein (1997) first formulated this *modified multiself consistency* notion of incentive compatibility, and established it as an implication of ex ante optimality.

[11]Our proof is necessarily more involved. The first step is standard, showing that the derivative of Dug's payoff in a given transition chance is proportional to the continuation payoff gain from this transition. But Dug doesn't directly choose transition chances: He chooses thresholds, each of which impacts many transition chances, along with indirectly affecting payoffs through changes in post-amnesia beliefs.

A memory policy is *interior* if it uses every memory state, i.e. $0 < p_1 < \cdots < p_{N-1} < 1$. We use the first Proposition 1 result to establish (Appendix B.1) that optimal memory policies are interior; and thus, that Dug's payoff strictly increases in the number of memory states.

Corollary 1 *Optimal memory policies are interior, so Dug's value strictly increases in N.*

Finally, we establish that Bayesian consistent beliefs are well-defined. As explained following (9), this is complicated by the fact that we need to know Dug's beliefs to determine his long-run distribution ρ^θ over memory states, and in turn need this distribution to compute beliefs. Thus, the Bayesian consistent belief vector q is a fixed point rather than a simple formula. An additional complication is that existing papers have focused on very small termination chances, where jumps to non-adjacent memory states were suboptimal (so that the transition matrix was mostly zeros). But Dug can transition to *any* memory state between amnesia shocks while he is a Bayesian rational learner. Nonetheless, our model has enough structure to compute beliefs:

Proposition 2 (Post-Amnesia Beliefs) *There exist unique post-amnesia beliefs $(q_n)_{n=1}^N$ satisfying* (9) *for any memory policy* $\mathbf{p}$*. Moreover, these post-amnesia beliefs and the associated conditional distributions over memory states ρ^θ are continuous in memory thresholds p_n; and for all n, the belief q_n lies inside the memory state n interval $[p_{n-1}, p_n]$.*

Our proof in Appendix C is constructive, with a recursive algorithm that pins down unique beliefs $(q_1, q_2, \ldots, q_N)$ for any interval policy $\mathbf{p}$. Our recursion solves for all q_n "from the outside-in." The probability mass leaving any block of memory states $\{1, 2, ..., n\}$ must match the probability mass entering this block, but we use (6) to show that the latter probability has the same likelihood ratio (in state H compared to L) from *any* memory state $k > n$. This permits a formula for beliefs in state $n < n_0$ that does not depend on beliefs in higher memory states. A symmetric algorithm solves from the top down for beliefs in memory states above n_0, and finally q_{n_0} depends on the average belief in states above and below n_0. Continuity follows trivially from the continuity of every implicit function in our recursion, along with continuity of the transition chances (6).

5 Illustrative Special Case with $N = 2$ Memory States

A. Optimality Conditions and Comparative Statics. When $N = 2$, Dug chooses a single threshold belief p_1 along with beliefs q_1 and q_2 in his two memory states. Fundamentally, this example will reduce to solving three equations in these three unknowns. We now see how these equations arise. Since the prior bias favors state $\theta = H$, i.e. $\mu\Pi \geq \pi$, Dug should start in memory state 2, since this biases the memory toward action $\mathcal{H}$.

Using (11), Dug wishes to maximize $\sum_{n=1}^{2}(\mu\rho_n^H w_n^H + \rho_n^L w_n^L)$, with w_n^θ given by (10). By (8), the probabilities ρ_n^θ of memory states $n = 1, 2$ at the final amnesia shock are steady-state frequencies of a perturbed Markov process, where Dug jumps to memory state $n_0 = 2$ with chance η, otherwise transitions according to $\lambda_{i,j}^\theta$:

$$\rho_1^\theta = \frac{(1-\eta)\lambda_{2,1}^\theta}{\eta + (1-\eta)\lambda_{1,2}^\theta + (1-\eta)\lambda_{2,1}^\theta} \text{ and } \rho_2^\theta = \frac{\eta + (1-\eta)\lambda_{1,2}^\theta}{\eta + (1-\eta)\lambda_{1,2}^\theta + (1-\eta)\lambda_{2,1}^\theta} \tag{14}$$

But by (5) along with (6), transition chances in $\theta = H$ (swap $\xi + 1$ and $\xi - 1$ for $\theta = L$) from Dug's rational learning between shocks are as follows, with $P_1 \equiv p_1/(1-p_1)$ and $Q_n \equiv q_n/(1-q_n)$:

$$\lambda_{1,2}^H = \frac{\xi+1}{2\xi}\left(\frac{Q_1}{P_1}\right)^{\frac{1}{2}(\xi-1)} \text{ and } \lambda_{2,1}^H = \frac{\xi-1}{2\xi}\left(\frac{P_1}{Q_2}\right)^{\frac{1}{2}(\xi+1)} \tag{15}$$

Notice that Dug's payoff directly depends on his post-amnesia beliefs Q_1 and Q_2, as the transition chances (15) do, thus so does ρ_n^θ from (14) and w_n^θ from (10). So conceivably, something other than Bayes consistency could be optimal. But our main technical results in Lemma 1 and Proposition 1 find two *necessary* conditions for optimality. First, Bayesian beliefs: Dug's beliefs are Bayesian consistent (obey (9)) if $Q_n = \mu\rho_n^H/\rho_n^L$ for $n = 1, 2$. In Appendix D, we simplify this as follows, where $\underline{x}$ (the root of function f in (38)) depends only on parameters η and ξ:

$$Q_1 = P_1\underline{x}, \text{ and } \left(\frac{Q_1}{Q_2}\right)^{\frac{\xi-1}{2}} \frac{\mu - Q_1}{Q_2 - \mu} = \frac{\eta}{1-\eta}\frac{\xi-1}{2}\underline{x}^{\frac{\xi-1}{2}}\left(\frac{\xi+1}{\xi-1} - \underline{x}\right) \tag{16}$$

Second, Dug must be indifferent between expected post-amnesia payoffs in memory states 1 and 2 at the threshold between them. From Appendix D, this is:

$$\mu = \frac{\xi+1}{\xi-1}\frac{\Pi}{\pi}Q_1Q_2\left(\frac{1 - \frac{\xi+1}{2\xi}\left(\frac{\Pi Q_1}{\pi}\right)^{\frac{\xi-1}{2}} - \frac{\xi-1}{2\xi}\left(\frac{\pi}{\Pi Q_2}\right)^{\frac{\xi+1}{2}}}{1 - \frac{\xi-1}{2\xi}\left(\frac{\Pi Q_1}{\pi}\right)^{\frac{\xi+1}{2}} - \frac{\xi+1}{2\xi}\left(\frac{\pi}{\Pi Q_2}\right)^{\frac{\xi-1}{2}}}\right) \tag{17}$$

While these conditions are only necessary, we prove in Appendix D that they have a unique solution, and thus are also sufficient to fully characterize the $N = 2$ optimum. We also derive the following comparative statics:

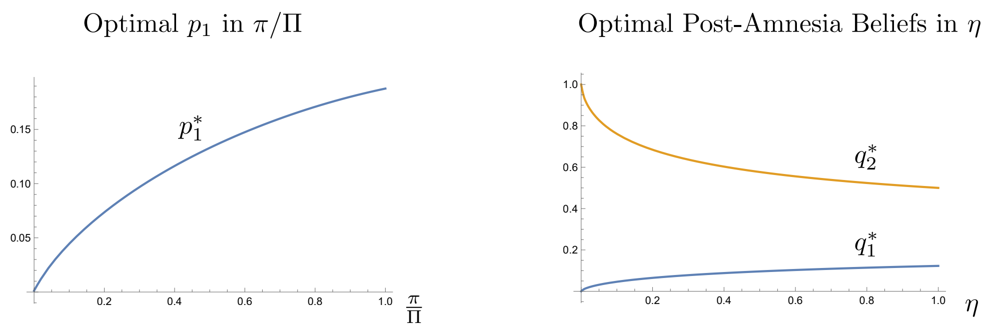

Figure 1: **Optimal Two State Memory.** The left graph depicts the optimal interval cutoff p_1 as a function of the payoff ratio $\pi/\Pi \leq 1$ for $N = 2$ and $\mu = \sigma = \alpha = \delta = 1$. The right graph depicts the optimal post-amnesia beliefs q_1 and q_2 for $\mu = 1$, $\pi/\Pi = 1/2$, and $\xi = 2$.

Proposition 3 (Sufficiency and Comparative Statics) *When $N = 2$, the Bayesian consistency* (16) *and indifference* (17) *equations are necessary and sufficient to determine the optimal threshold p_1. This threshold rises with π/Π and with μ. As η rises, the associated optimal beliefs move closer together, with Q_1 rising and Q_2 falling.*

The comparative static for π/Π is intuitive; as this ratio rises, i.e. matching the high state becomes relatively less important, the high memory state interval shrinks (p_1 rises). The μ comparative static may seem counterintuitive, finding that this also happens as the high state becomes **more** likely. The countervailing factor is that higher μ raises post-amnesia beliefs in both memory states, which make transitions into memory state 2 more likely. The net impact is indeed more time in memory state 2, via a threshold that rises, but proportionately less than μ. To see this, consider increasing μ, and as a thought experiment, also increase π/Π so as to hold constant the overall bias $\beta \equiv \mu\Pi/\pi$. Notice that with β unchanged, both (16) and (17) remain satisfied if we hold constant Q_1/μ, Q_2/μ, and P_1/μ. This leaves unchanged the transition chances in (6), and thus also the relative frequencies of memory states 1 and 2. But now, reduce π/Π to its actual value; this reduces the threshold, by Proposition 3, increasing the frequency of memory state 2. In the end, as is intuitive, Dug spends more time in the high memory state, and post-amnesia beliefs rise but by proportionally less than μ.

We highlight another way our continuous model differs from the literature. With discrete learning, there is little gain to asymmetry in both the prior μ and the action bias Π/π: Both simply affect the relative importance of matching states $\theta = H, L$, with identical impacts. But for Dug, the action bias also affects the terms w_n^θ from (10) (which affect his payoff (11)), while the prior impacts Bayesian consistent beliefs. Thus they have different effects, even holding fixed the overall bias β towards the high state. In particular, consider doubling μ and halving Π. While we show in Appendix C that for a *given* interval policy, only Q_{n_0} depends directly on μ, we just argued that the *optimal* policy adjusts so that all likelihood ratios rise proportionately with μ.

Imagine that Dug is driving, but prone to distraction shocks (looking at his iPhone). In the Low state, there's no traffic and no evasive action needed (doing nothing gains $\pi > 0$ compared to needlessly slamming on the brakes). In the High state, a random decision shock in the form of stopped traffic will arise, and Dug gains a large amount $\Pi > 0$ by taking action to avoid crashing. In the discrete literature, more significant crash damages (large Π) and higher chances of stopped traffic (large μ) have symmetric impacts. But for Dug, doubling μ while halving Π both doubles his perceived accident risk in both memory states, and doubles the threshold to switch to his more vigilant memory state. The fraction of time spent in his "high-caution" memory state 2 remains unchanged, but he's more cautious in both.

6 Values as a Function of Current Beliefs

A. Conditional Values. We now explore Dug's continuous time value as a function of beliefs, given true state θ. Fix an interval policy $\mathbf{p}$, and let $\lambda^{\theta}_{n,k}$ be the memory state transitions from (6) assuming Bayesian consistent post-amnesia beliefs.

Define the log-likelihood memory state thresholds $\ell_n = \log(p_n/(1-p_n))$, the cutoff between optimal terminal actions $\hat{\ell} \equiv \log(\hat{p}/(1-\hat{p})) = \log(\pi/\Pi)$, and the log prior likelihood ratio $\hat{\mu}$. Let $V^{\theta}(\ell)$ be the expected payoff given current log-likelihood ratio ℓ conditional on state θ. Given post-amnesia values ν^{θ}_n (from (12)), these conditional values obey:

$$V^L(\ell) = \eta\pi F^L(\hat{\ell}-\ell) + (1-\eta)\sum_n \lambda^L_n(\ell)\nu^L_n \tag{18}$$

$$V^H(\ell) = \eta\Pi\left(1-F^H(\hat{\ell}-\ell)\right) + (1-\eta)\sum_n \lambda^H_n(\ell)\nu^H_n \tag{19}$$

where we overuse notation in defining the chance that the next shock hits in memory state n given current log-likelihood ratio ℓ as

$$\lambda^{\theta}_n(\ell) \equiv F^{\theta}(\ell_n-\ell) - F^{\theta}(\ell_{n-1}-\ell)$$

Lemma 2 *Given any interval policy, Dug's expected payoffs* $(\nu^{\theta}, V^{\theta})$ *are the unique solutions to* (12), (18), *and* (19) *evaluated at the unique post-amnesia beliefs* $\mathbf{q}$ *satisfying* (9) *with associated transitions* λ^{θ} *given by* (6). *These values are* C^1 *everywhere,* C^2 *on each open interval* (ℓ_{n-1}, ℓ_n), *and strictly monotone with* (ν^L, V^L) *decreasing and* (ν^H, V^H) *increasing.*

PROOF: We prove the case of $\theta = L$; the logic for $\theta = H$ is symmetric. The fact that V^L is C^1 everywhere and C^2 on each open interval (ℓ_{n-1}, ℓ_n) follows from (18) and the fact that $F^\theta(x)$ from (5) is everywhere C^1 in x (*even at* $x = 0$), and C^2 for $x \neq 0$.

To see that ν_n^L strictly decreases in n, it suffices to show that this property is preserved by the operator T in (12). To this end, recall that post-amnesia log likelihood ratios are ordered $\ell_1 < \ell_2 < \cdots < \ell_N$. This implies that the conditional distribution over memory states k at the next amnesia shock, $\lambda_{n,k}^L$, from (6), is first order increasing in n. In addition, w_n^θ from (10) strictly decreases in n. Thus Tv_n^L is strictly decreasing in n for any vector of post-amnesia values that are non-increasing in n. Thus, the fixed point post-amnesia values obey:

$$\nu_1^L > \nu_2^L > \cdots > \nu_N^L \tag{20}$$

To show that V^L is strictly decreasing, we differentiate (18) and rearrange to discover:

$$(V^L)'(\ell) \quad = \quad = -\eta\pi f^L(\hat{\ell} - \ell) - (1-\eta)\sum_{n=1}^{N-1} f^L(\ell(p_n) - \ell)\left(\nu_n^L - \nu_{n+1}^L\right) < 0$$

□

B. Unconditional Values. Define $V_n(p) \equiv pV^H(\ell(p)) + (1-p)V^L(\ell(p))$ as Dug's unconditional payoff when his belief p is in memory state n. This obeys the following Hamilton-Jacobi-Bellman (HJB) equation, where $u(p) = \max\{p\Pi, (1-p)\pi\}$ is his terminal payoff at belief p:

$$0 = \delta\left(u(p) - V_n(p)\right) + \alpha\left(p\nu_n^H + (1-p)\nu_n^L - V_n(p)\right) + 2\frac{p^2(1-p)^2}{\sigma^2}V_n''(p) \tag{21}$$

The RHS of (21) is the expected drift in the value: The first term reflects the δ chance that the problem ends, replacing value $V_n(p)$ by terminal payoff $u(p)$. The second term reflects the α chance of a distraction shock, in which case the value reverts to post-amnesia payoff $p\nu_n^H + (1-p)\nu_n^L$. And the final term reflects expected drift in his value due to volatility of the Gaussian learning process. This rearranges to:

$$V_n(p) = \eta u(p) + (1-\eta)\nu_n(p) + \frac{2p^2(1-p)^2}{(\alpha+\delta)\sigma^2}V_n''(p) \tag{22}$$

The sum of the first two terms is Dug's expected value if his beliefs never change, assuming terminal reward $\nu_n(p)$ following a distraction shock, and the final term is the normalized flow *value of the information* he extracts from observing the signal process (1).

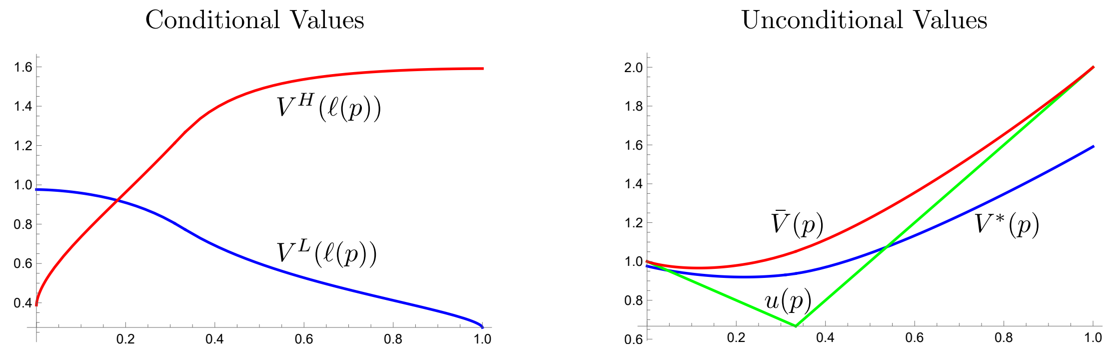

Figure 2: **Value Functions.** Left: conditional value functions $V^\theta(\ell(p))$ given an optimal memory policy. Right: the optimal unconditional value $V^*(p)$, the expected terminal payoff $u(p)$, and the value with no distractions (red). Here: $\Pi = N = 2$ and $\pi = \alpha = \delta = \sigma = 1$.

Lemma 3 (Smooth Pasting) *For any memory policy, the unconditional value V is C^1 everywhere, and C^2 and strictly convex on each open interval (p_n, p_{n-1}).*

PROOF: That V is C^1 everywhere and C^2 on each (p_n, p_{n-1}) follows from V^L and V^H each C^1 (Lemma 2). We prove strict convexity in Appendix E. $\square$.

C. Smooth and Super Smooth Pasting. Rephrasing Lemma 3, for *any* interval policy, the unconditional value function obeys value matching and smooth pasting across all barriers. In other words, unlike in standard stochastic control models with a linear state space, e.g., Dixit (2013), smooth pasting is not an optimality condition.[12] It simply follows from the fact that cdf F^θ in (5) is continuous and differentiable, even at zero, and thus, so are transition chances between amnesia shocks. Intuitively, the difference from standard control models is that Dug doesn't immediately take an action as he crosses the boundary. He must wait for nature to offer him another shock, and so switching memory states doesn't introduce any discontinuities into Dug's eventual action choice.

This begs the question of what is the optimality condition. By Lemma 3, the unconditional value function V_n is C^2 on open intervals (p_{n-1}, p_n). *Super Smooth Pasting (SSP)* imposes continuity of the second derivative at the boundary between memory states:

$$\text{SSP:} \quad V_n''(p_n) = V_{n+1}''(p_n) \tag{23}$$

By (22) SSP is equivalent to continuity of the value of information at p_n.

[12]Potentially, the value function could violate these conditions with a non-interval policy, or an interval policy where the vector of thresholds varies with where amnesia last struck. As noted below Proposition 1, we considered the latter generalization in Appendix B and proved that it does not help

It turns out that SSP is equivalent to Dug's indifference between being hit with a distraction shock just to the left or right of the boundary between memory regions.

Lemma 4 (Super Smooth Pasting) *SSP obtains at threshold p_n if and only if indifference condition* (13) *obtains.*

PROOF: Substitute equation (22) into $V_n(p_n) = V_{n+1}(p_n)$ (by continuity) to get:

$$(1-\eta)\nu_n(p_n) + \frac{2p_n^2(1-p_n)^2}{\alpha+\delta}V_n''(p_n) = (1-\eta)\nu_{n+1}(p_n) + \frac{2p_n^2(1-p_n)^2}{\alpha+\delta}V_{n+1}''(p_n)$$

and thus, Dug is indifferent across memory states n and $n+1$ at threshold $n+1$, if and only if his instantaneous value of information is continuous at p_n. More generally:

$$\nu_n(p_n) \gtreqless \nu_{n+1}(p_n) \quad \Leftrightarrow \quad V_n''(p_n) \lesseqgtr V_{n+1}''(p_n) \qquad \square$$

Proposition 1 established that optimality implies Dug's indifference at the thresholds p_n. Thus, SSP is a necessary optimality condition expressed in terms of the unconditional value.[13]

D. Distraction and Decision Capital Gains and Losses. Let $V_n^*(p)$ be the *optimized* unconditional value at belief p in memory state n, namely $V_n(p)$ given an optimal memory policy. WE now explore how this value is affected by the shocks Dug experiences. Distraction shocks are a capital loss, erasing knowledge that Dug has accumulated. But more surprisingly, decision shocks can actually be a capital *gain* when Dug is at his most confident:

Proposition 4 *Distraction shocks are always a capital loss: for all memory states n and beliefs p, $V_n^*(p) \geq p\nu_n^H + (1-p)\nu_n^L$. But decision shocks are not: There exist $\underline{p} < p_1$ and $\overline{p} > p_{n-1}$ such that $V^*(p) > u(p)$ for $p \in (\underline{p}, \overline{p})$, and (b) $u(p) > V^*(p)$ for $p \notin [\underline{p}, \overline{p}]$.*

The proof is in Appendix E. For beliefs sufficiently close to 0 and 1, Dug would strictly benefit if he could stop learning and take immediate action: His myopic payoff $u(p)$ exceeds his value from continuing. He rationally forecasts that his future distraction shocks will harm him, and potentially pull him far from his current strong belief.[14] Figure 2 illustrates many of our results for value functions: namely, the monotonicity of conditional values V^θ (Lemma 2),

[13] This is in contrast to Dumas (1991), where smooth pasting and super smooth pasting were joint optimality conditions for the barrier in a regime shift model. Notice that while F^θ from (5) is C^1, making value matching and smooth pasting automatic, its *derivative* is not, with a kink at zero.

[14] This did not happen in Wilson (2014). Indeed, an early version of that paper considered allowing the decision-maker to choose when to stop, but it had little effect: She was indifferent about quitting in the extremal memory states, but never strictly gained from it.

the convexity of the optimal unconditional value V^* (by Lemmas 3 – 4 and Proposition 1), and the relationship between the optimal unconditional value and the expected stopping payoff $u(p)$ described in Proposition 4. The right panel illustrates (for $N = 2$) that for confident beliefs, Dug's myopic payoff (green) exceeds his optimized value with distractions (blue), while the gap between his value (blue) and his value with no distractions (red) depicts his value lost due to distractions.

7 Value Comparative Statics

As seen in Proposition 4, Distraction shocks always induce a capital loss, but decision shocks can be bad or good news at the moment they occur. The next result asserts that Dug's ex ante expected payoff falls as decision shocks become more common.

Proposition 5 *Dug's optimal initial value $\mathcal{V}^*$ is strictly decreasing in δ and σ.*

The proof is in Appendix F. Intuitively, increases in σ make the observation process more noisy, while increasing δ reduces the expected time that Dug has to acquire information before taking an action.

We now derive the limit behavior of the optimal initial value, emphasizing the dependence on N by writing $\mathcal{V}^*(N)$. We first explicitly solve for $\mathcal{V}^*(1)$: With just one memory state, its post-amnesia belief must be $q_1 = \mu$, and so, by (10) and (11),

$$\mathcal{V}^*(1) = \mu\nu_1^H + \nu_1^L = \mu\Pi\left(1 - \frac{(\xi-1)}{2\xi}\left(\frac{\pi}{\mu\Pi}\right)^{\frac{1+\xi}{2}}\right) + \frac{\pi(1+\xi)}{2\xi}\left(\frac{\pi}{\mu\Pi}\right)^{\frac{\xi-1}{2}} = \mu\Pi + \frac{\pi}{\xi}\left(\frac{\pi}{\mu\Pi}\right)^{\frac{\xi-1}{2}}$$

Now, define the *value with unbounded memory*, $\mathcal{V}^*(\infty) \equiv \lim_{N\to\infty} \mathcal{V}^*(N)$ and the *full information value* $\mathcal{V}^{FI} = \mu\Pi + \pi$. We then have the following limits.

Proposition 6 (Limit Values) *The value with unbounded memory is*

$$\lim_{\alpha\to 0} \mathcal{V}^*(N) = \mathcal{V}^*(\infty) = \mu\Pi + \frac{\pi}{\zeta}\left(\frac{\pi}{\mu\Pi}\right)^{\frac{\zeta-1}{2}} \qquad \text{where} \quad \zeta = \sqrt{1+2\delta\sigma^2}$$

The full information value emerges as $\sigma \to 0$, or as $\delta \to 0$ provided $N \geq 2$. We also have:

$$\lim_{\alpha\to\infty} \mathcal{V}^*(N) = \lim_{\delta\to\infty} \mathcal{V}^*(N) = \lim_{\sigma\to\infty} \mathcal{V}^*(N) = \mu\Pi$$

The first of these results highlights that with no amnesia shocks, Dug is a standard rational Bayesian, and thus earns the payoff he would obtain with infinite memory. The

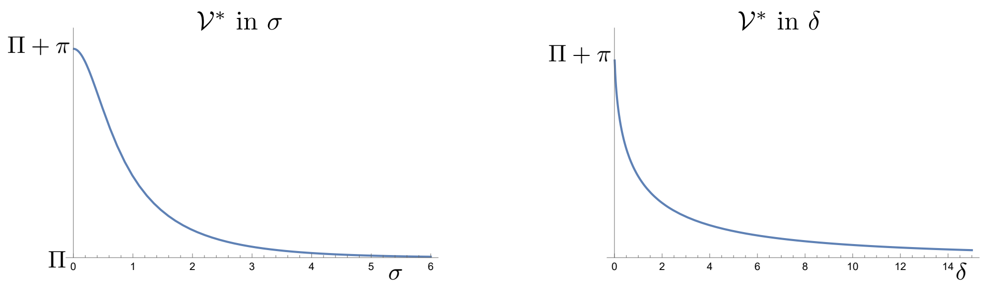

Figure 3: **Changes in the Initial Value.** Each graph illustrates how the optimal initial value $\mathcal{V}^*$ varies in σ (left) and δ (right). In all graphs the y-axis origin is Π, $\pi/\Pi = 1/2$, and all remaining parameters are set to 1, e.g., in the left graph $\alpha = \delta = 1$.

formula simply evaluates $\mathcal{V}^*(1)$ at $\alpha = 0$ (where ξ becomes ζ), since with no amnesia shocks, it doesn't matter how many memory states Dug has. The second highlights that two memory states are sufficient to achieve the full information value as $\delta \to 0$. Intuitively, when decision shocks are vanishingly rare, he knows that he will observe the signal process for a near infinite length of time before making a decision. Thus, he can be nearly certain that $\theta = L$ in memory state 1 and that $\theta = H$ in memory state 2, and both states are nearly absorbing. But two memory states are also necessary to achieve the full information value, since evaluating $\mathcal{V}^*(1)$ at $\delta = 0$ gives a lower payoff:

$$\lim_{\delta \to 0} \mathcal{V}^*(1) = \mu\Pi + \frac{\pi}{\kappa}\left(\frac{\pi}{\mu\Pi}\right)^{\frac{\kappa-1}{2}} < \mathcal{V}^{FI} \quad \text{for } \kappa = \sqrt{1 + 2\alpha\sigma^2}$$

Figure 3 illustrates Propositions 5 and 6.

8 The Demand for Memory and Information

A. The Demand for Memory. Assume that at time 0 Dug can choose the number of memory slots N at increasing additive cost $C(N)$ with $C(1) = 0$ and $\lim_{N\to\infty} C(N) > \pi$. To emphasize the dependence on N and the rate of amnesia shocks α, write the optimal initial value (11) as $\mathcal{V}^*(N, \alpha)$, so that Dug's optimal memory correspondence obeys:

$$N^*(\alpha) \equiv \arg\max_{N\in\mathbb{N}} [\mathcal{V}^*(N, \alpha) - C(N)] \tag{24}$$

This problem must have a finite solution. Indeed, Dug's value is bounded below by $\mu\Pi$, since he can always secure this value by selecting action $\mathcal{H}$ whenever the decision shock hits, and his value is bounded above by the full information value $\pi + \mu\Pi$. Consequently, the gain from buying infinite memory is bounded above by π, which is strictly below $\lim_{N\to\infty} C(N)$.

By standard comparative statics reasoning (Milgrom and Shannon (1994)) the demand

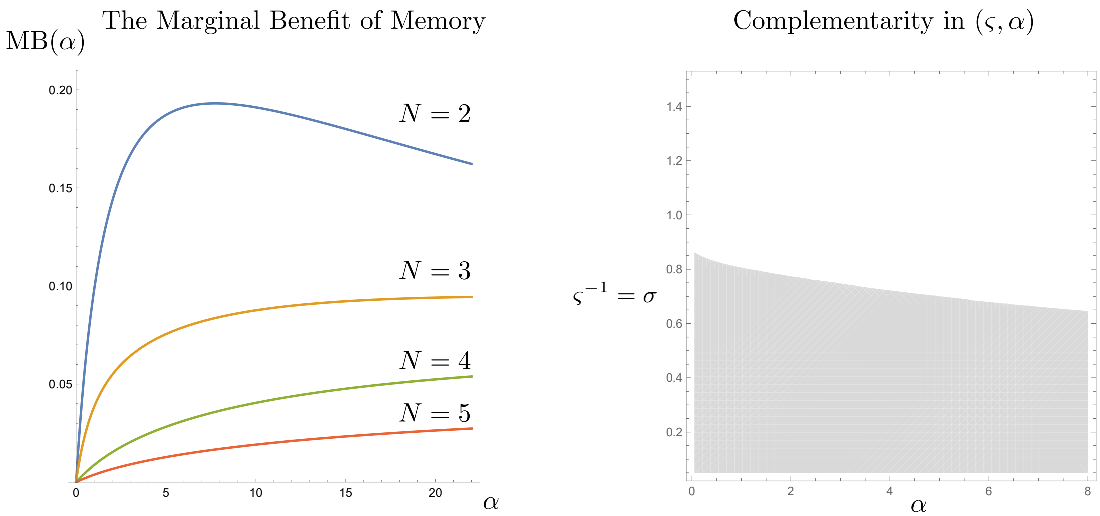

Figure 4: **Precision and Memory.** The left panel graphs the marginal benefit of memory as a function of the rate of amnesia shocks for $\delta = \sigma = 1$. The rate of amnesia shocks α (x-axis) and precision of the signal ς (y-axis) are complements on the shaded region and substitutes elsewhere (left graph) for $N = 2$ and $\delta = 1$.

for memory $N^*(\alpha)$ will be non-decreasing (non-increasing) for *any* cost function when the marginal benefit of an additional memory slot $\text{MB}(N, \alpha) \equiv \mathcal{V}^*(N+1, \alpha) - \mathcal{V}^*(N, \alpha)$ is increasing (decreasing) in α. Furthermore, monotonicity of the marginal benefit is necessary for memory demand to be monotone in α for all cost functions. Unfortunately, Propositions 5 and 6 together imply that the MB cannot be monotonic in α at any N.

Corollary 2 *The marginal benefit of memory is strictly positive for all* $0 < \alpha < \infty$ *with limits* $\lim_{\alpha \to 0} MB(N, \alpha) = \lim_{\alpha \to \infty} MB(N, \alpha) = \lim_{N \to \infty} MB(N, \alpha) = 0$.

In other words, the MB of memory is initially rising in α, and vanishes in N and α.

The simplest case consistent with the limits in Corollary 2 is that the marginal benefit of memory is decreasing in N and hump-shaped in α. Under these assumptions, the optimal number of memory slots $N^*(\alpha)$ is hump-shaped in α for any convex cost function $C(N)$. While we have not been able to establish these properties analytically, they do hold in every computed example. Figure 4 (left) plots the MB of memory for four values of N.

B. The Demand for Information. Parameterize information by the precision of the signal process, $\varsigma \equiv \sigma^{-1}$. To emphasize the dependence on ς and the rate of amnesia shocks α, write the optimal initial value as $\mathcal{V}^*(\varsigma, \alpha)$. Assume that Dug chooses precision at increasing cost $C(\varsigma)$ with $C(0) = 0$ and $\lim_{\varsigma \to \infty} C(\varsigma) > \pi$. The optimal precision correspondence is thus:

$$\varsigma^*(\alpha) \equiv \arg\max_{\varsigma \geq 0} \left[\mathcal{V}^*(\varsigma, \alpha) - C(\varsigma)\right] \tag{25}$$

This problem must have a finite solution, by identical reasoning to that following (24).

As above, the demand for precision $\varsigma^*(\alpha)$ will be non-decreasing (non-increasing) for *any* cost function when the difference $\mathcal{V}^*(\varsigma,\alpha'') - \mathcal{V}^*(\varsigma,\alpha')$ is increasing (decreasing) in ς for all $\alpha'' > \alpha'$. Proposition 6 yields the following limits for such differences.

Corollary 3 *The change in value from a change in the amnesia shock rate obeys*

$$\lim_{\varsigma\to 0}\left[\mathcal{V}^*(\varsigma,\alpha'') - \mathcal{V}^*(\varsigma,\alpha')\right] = \lim_{\varsigma\to\infty}\left[\mathcal{V}^*(\varsigma,\alpha'') - \mathcal{V}^*(\varsigma,\alpha')\right] = 0$$

Thus, the difference $\mathcal{V}^*(\varsigma,\alpha'') - \mathcal{V}^*(\varsigma,\alpha')$ cannot be strictly monotone in ς. Equivalently, $\mathcal{V}^*$ cannot be globally supermodular or globally submodular. The simplest intuitive possibility consistent with these limits is that $\mathcal{V}^*$ is submodular for low levels of precision and supermodular for high levels of precision. This is true in all of our computed examples. In particular, for $N = 2$ and $\delta = 1$ we computed $\mathcal{V}^*$ on a fine grid of parameter values $(\varsigma_0, \varsigma_1, ...\varsigma_K) \times (\alpha_0, \alpha_1, ...\alpha_K)$ and then computed the cross partial differences:

$$\mathcal{V}^*(\varsigma_{i+1}, \alpha_{i+1}) \approx \mathcal{V}^*(\varsigma_{i+1}, \alpha_{i+1}) + \mathcal{V}^*(\varsigma_i, \alpha_i) - \mathcal{V}^*(\varsigma_{i+1}, \alpha_i) - \mathcal{V}^*(\varsigma_i, \alpha_{i+1})$$

Figure 4 illustrates the regions on which these cross partial differences are positive (shaded) and negative (white) in $(\alpha, \sigma = \varsigma^{-1})$ space. Notice that for any fixed α, there is a threshold value of precision, such that the cross partial differences are positive above this precision (low values of σ) and negative below this level of precision.[15] Thus, for these parameter values, Dug chooses more precision (i.e. better information) as amnesia shocks become more common when precision is already high (aka when precision is cheap), but chooses worse information when amnesia shocks become more common when precision is already low.

9 Conclusion

Our model is the first to formally investigate how distractions impact rational Bayesian learning, and how they might optimally be mitigated. While this setup introduces some technical complications, e.g., the fact that post-distraction beliefs directly impact the payoff, it is tractable for all parameters, and delivers comparative statics not found in the discrete literature. One novel finding is that at very confident beliefs, the anticipated impact of future distractions is so significant that Dug would actually prefer to stop learning. This does not happen in typical learning models, but suggests an interesting possibility for future research.

We note that post-amnesia beliefs between distraction shocks are not a martingale. Clearly, they must drift up from memory state 1 and down from memory state N. But

[15] The same super(sub)modularity pattern obtains for all values of δ we have numerically tested.

this is the only definitive pattern we found, and in particular it is not the case that there is a cutoff memory state such that Dug expects his beliefs to drift up below the cutoff, and down above the cutoff. For some intuition, note that raising n (and hence q_n) means fewer memory states $k > n$ that entail a positive drift $q_k - q_n$ (and more states $k < n$ with negative drift), while also decreasing the magnitude of each drift term. This effect – it's harder to drift up when beliefs are already high – reduces expected drift as n rises. The competing effect is that increasing n also makes transitions to higher memory states more likely, which increases expected drift.

While our formal motivation is that of an individual decision maker learning in the face of distractions, the same model could describe a sequence of decision makers with random transition times between them. For example, each DM in the sequence could be a worker within a firm in a given position (e.g., a lead researcher on an R&D project) with "distraction" shocks separating the worker from the firm. Or our model could capture a sequence of doctors seeking to diagnose a patient, where poor communication – both in the form of inadequate notes and excessive notes – have often been blamed for poor patient outcomes.[16]

[16] For example, see Steiner, Stewart, and Matejka (2019).

A Derivation of Belief Evolution (4) and (5)

Step 1 *Deriving the Ito Process for Conditional Beliefs* (4)

Lemma 1 in Anderson and Smith (2013) derives the following Ito diffusion for an uninformed player's beliefs (here, Dug) when observing a signal process with state contingent drift:

$$d^\theta p = \begin{cases} 4\sigma^{-2}p(1-p)^2 dt + 2\sigma^{-1}p(1-p)dW & \text{if } \theta = H \\ -4\sigma^{-2}p^2(1-p)dt + 2\sigma^{-1}p(1-p)dW & \text{if } \theta = L \end{cases} \tag{26}$$

Applying Ito's Lemma to (26) with $\ell(p) = \log(p/(1-p))$, and simplifying using $\ell'(p) = [p(1-p)]^{-1}$ and $\ell''(p) = (2p-1)[p^2(1-p)^2]^{-1}$, completes the derivation of (4).

Step 2 *The derivation of the conditional distribution at the next shock* (5).

In state θ, the CDF $\Phi^\theta(x,t) \equiv \Pr(\ell - \ell_0 \leq x|\theta)$ is given by

$$\begin{aligned} \Phi^\theta(x,t) &= \frac{1}{2} + \frac{1}{2}\text{erf}\left(\frac{x-\mu_1}{\sigma_1\sqrt{2}}\right), \quad \text{with } \mu_1 = \frac{2ts_\theta}{\sigma^2} \quad \text{and} \quad \sigma_1 = \frac{2\sqrt{t}}{\sigma} \\ &\Rightarrow \quad \Phi^H(x,t) = \frac{1}{2} + \frac{1}{2}\text{erf}\left(\frac{\sigma}{2}\frac{x - \frac{2t}{\sigma^2}}{\sqrt{2t}}\right) \text{ and } \Phi^L(x,t) = 1 - \Phi^H(-x,t) \end{aligned} \tag{27}$$

In state H, denote $\phi^H(x,t) \equiv \partial\Phi^H/\partial x$ and $\psi^H(x,t) \equiv \partial\Phi^H/\partial t$. These are given by:

$$\phi^H(x,t) = \frac{\sigma}{2\sqrt{2\pi t}}\exp\left(-\frac{\sigma^2}{8t}\left(x - \frac{2t}{\sigma^2}\right)^2\right) \quad \text{and} \quad \psi^H(x,t) = -\frac{1}{2t}\left(x + \frac{2t}{\sigma^2}\right)\phi^H(x,t)$$

The chance that the next distraction hits at time t, given no intervening decision shock, is:

$$\Pr(\text{amnesia at } t|\text{amnesia before decision}) = \frac{\alpha e^{-\alpha t}\int_t^\infty \delta e^{-\delta s}ds}{\frac{\alpha}{\alpha+\delta}} = (\alpha+\delta)e^{-(\alpha+\delta)t} \tag{28}$$

And thus, the chance that $\ell - \ell_0$ changes by at most x at the next amnesia shock is as follows:

$$\begin{aligned} F^H(x) &= \int_0^\infty (\alpha+\delta)e^{-(\alpha+\delta)t}\Phi^H(x,t)dt \\ &= -\lim_{t\to\infty}\Phi^H(x.t)e^{-(\alpha+\delta)t} + \lim_{t\to 0}\Phi^H(x,t)e^{-(\alpha+\delta)t} + \int_0^\infty e^{-(\alpha+\delta)t}\psi^H(x,t)dt \\ &= 0 + 1_{x\geq 0} - \frac{\sigma}{4\sqrt{2\pi}}\int_0^\infty e^{-(\alpha+\delta)t}t^{-\frac{3}{2}}\left(x + \frac{2t}{\sigma^2}\right)\exp\left(-\frac{\sigma^2}{8t}\left(x - \frac{2t}{\sigma^2}\right)^2\right)dt \end{aligned}$$

(the second line integrated by parts with $u = \Phi^H(x,t), dv = (\alpha+\delta)e^{-(\alpha+\delta)t}, du = \psi^H(x,t)$, $v = -e^{-(\alpha+\delta)t}$). Evaluate this integral and simplify with $\xi = \sqrt{1+2(\alpha+\delta)\sigma^2}$ to recover (5).

B Proof of Lemma 1 and Proposition 1

Begin with a generalized policy, where the vector of thresholds $\mathbf{p_i}$ can depend on the memory state i where amnesia last struck. We work with likelihood ratios $Q_i = q_i/(1-q_i)$ and $P_{i,j} = p_{i,j}/(1-p_{i,j})$. Recall by (11) that Dug wishes to maximize $\mathcal{V} = \mu\mathcal{V}^H + \mathcal{V}^L$, with

$$\mathcal{V}^\theta = \sum_{i=1}^{N} \mu\rho_i^\theta w_i^\theta,\ w_i^H = \Pi\left(1 - F^H\left(\log\frac{\pi}{\Pi Q_i}\right)\right),\ w_i^L = \pi F^L\left(\log\frac{\pi}{\Pi Q_i}\right) \tag{29}$$

The proof proceeds in 6 steps. Step 1 calculates the payoff impact of a change in Q_i via the w_i^θ terms. Step 2 calculates the payoff derivative in transition chances. Steps 3 and 4 derive the indifference FOC's for optimal **thresholds** (vs transition chances), allowing arbitrary beliefs; Step 3 simplifies the problem using cumulative transition chances, and Step 4 proves that optimally, $\mathbf{p_i}$ is the same $\forall i$. Step 5 shows that given the FOC's, Bayesian beliefs are optimal, proving Lemma 1. Step 6 simplifies the FOC's with Bayesian beliefs, proving Proposition 1.

Step 1 *The part of $\partial\mathcal{V}/\partial Q_i$ resulting from changes in w_i^θ has the same sign as $\mu\rho_i^H/\rho_i^L - Q_i$.*

The partial derivative of (29) in Q_i, considering only the w_i^θ terms, is $\mu\rho_i^H\partial w_i^H/\partial Q_i + \rho_i^L\partial w_i^L/\partial Q_i$. But by (29) along with (5), if $\pi > \Pi Q_i$ then this is

$$\begin{aligned} &\Pi\frac{\partial}{\partial Q_i}\left[\mu\rho_i^H\frac{\xi+1}{2\xi}\left(\frac{\Pi Q_i}{\pi}\right)^{\frac{\xi-1}{2}} + \rho_i^L\frac{\pi}{\Pi}\left(1 - \frac{\xi-1}{2\xi}\left(\frac{\Pi Q_i}{\pi}\right)^{\frac{\xi+1}{2}}\right)\right] \\ = \ &\Pi\frac{\xi+1}{2\xi}\frac{\xi-1}{2Q_i}\left(\frac{\Pi Q_i}{\pi}\right)^{\frac{\xi-1}{2}}\left[\mu\rho_i^H - Q_i\rho_i^L\right]\end{aligned}$$

As desired. A symmetric argument applies if instead $\pi < \Pi Q_i$. □.

Step 2 *Given constraint $\lambda_{i,i}^\theta = 1 - \sum_{j\neq i}\lambda_{i,j}^\theta$, the following holds $\forall j \neq i \in \{1, 2, ..., N\}$:* [17]

$$\frac{\partial\mathcal{V}^\theta}{\partial\lambda_{i,j}^\theta} = \rho_i^\theta\frac{1-\eta}{\eta}(\nu_j^\theta - \nu_i^\theta)$$

Fix an interval policy, and let $\Upsilon(\cdot)$ be the induced probability distribution over histories, with $\Upsilon(z_1|z_2)$ be the chance of history z_1 conditional on z_2 having occurred. Also recall that

[17] This step adapts the proof of Proposition 1 in Wilson (2014) (based on Piccione and Rubinstein (1997)).

$\Lambda^{\theta}_{i,j} \equiv (1-\eta)\lambda^{\theta}_{i,j}$ is the total transition chance $i \to j$ between amnesia shocks (given θ), with $\lambda^{\theta}_{i,j}$ the chance conditional on no decision shock. For any two memory states i and j, observe that $\Upsilon(z|\theta)$ can be decomposed as a term that does not depend on $\Lambda^{\theta}_{i,j}$ multiplied by $(\Lambda^{\theta}_{i,j})^{\#(z)}$, where $\#(z)$ denotes the number of occurrences of a transition $i \to j$ between amnesia shocks along the history z. Using this for the first equality below, and letting $\mathcal{H}(i \to j|z)$ be the set of all subhistories of z that end with an amnesia shock in memory state i followed by a transition to j by the next amnesia shock, we have

$$\frac{\partial}{\partial \Lambda^{\theta}_{i,j}}\,\Upsilon(z|\theta) = \#(z)\frac{\Upsilon(z|\theta)}{\Lambda^{\theta}_{i,j}} = \sum_{z' \in \mathcal{H}(i\to j|z)} \frac{\Upsilon(z|\theta)}{\Lambda^{\theta}_{i,j}} \tag{30}$$

Next, let $X(i)$ denote the set of all histories ending with an amnesia shock in state i, and for any history z, let $\Upsilon(z|(z',j),\theta)$ denote the probability of z conditional on z' followed by a transition (by the next amnesia shock) to state j, given θ. Then we have:

$$\sum_{z' \in \mathcal{H}(i\to j|z)} \frac{\Upsilon(z|\theta)}{\Lambda^{\theta}_{i,j}} = \sum_{z' \in \mathcal{H}(i\to j|z)} \Upsilon(z'|\theta)\Upsilon(z|(z',j),\theta) = \sum_{z' \in X(i)} \Upsilon(z'|\theta)\Upsilon(z|(z',j),\theta)$$

(This holds since the RHS summand vanishes whenever (z',j) is not a subhistory of z; and if it is—so z' (ending with an amnesia shock in i) followed by a transition $i \to j$ is a subhistory of z, and thus also $z' \in \mathcal{H}(i \to j|z)$—then in both the middle and RHS expressions, the corresponding term in the summand is equal to $\Upsilon(z|\theta)/\Lambda^{\theta}_{i,j}$). Plugging this into (30),

$$\frac{\partial}{\partial \Lambda^{\theta}_{i,j}}\Upsilon(z|\theta) = \sum_{z' \in X(i)} \Upsilon(z'|\theta)\Upsilon(z|(z',j),\theta)$$

Using this for the final equality in the first line below, we then obtain:

$$\begin{aligned}
\frac{\partial \mathcal{V}^{\theta}}{\partial \Lambda^{\theta}_{i,j}} &\equiv \frac{\partial}{\partial \Lambda^{\theta}_{i,j}} \sum_{j'=1}^{N} \sum_{z \in X(j')} \eta \Upsilon(z|\theta) w^{\theta}_{j'} = \eta \sum_{j'=1}^{N} \sum_{z \in X(j')} \left[\sum_{z' \in X(i)} \Upsilon(z'|\theta)\Upsilon(z|(z',j),\theta) \right] w^{\theta}_{j'} \\
&= \frac{1}{\eta} \sum_{z' \in X(i)} \eta \Upsilon(z'|\theta) \left(\sum_{j'=1}^{N} \sum_{z \in X(j')} \eta \Upsilon(z|(z',j),\theta)\, w^{\theta}_{j'} \right) \equiv \frac{1}{\eta}\rho^{\theta}_{i}\nu^{\theta}_{j}
\end{aligned}$$

(for the final simplification, the bracketed term is the payoff after a history z' followed by a transition to j; by stationarity this is $\nu^{\theta}_{j}\ \forall z'$). Multiply by $1-\eta$ for the derivative in $\lambda^{\theta}_{i,j}$; and subtract the derivative in $\lambda^{\theta}_{i,i}$ to incorporate constraint $\lambda^{\theta}_{i,i} = 1 - \sum_{j \neq i} \lambda^{\theta}_{i,j}$. □

Step 3 *Reformulating the Problem: The Indifference Conditions*

We reformulate the problem using cumulative transition chances. For any $i > j$, denote by $a^\theta_{i,j}$ the transition chance (between shocks) from i to $k \le j$; and for any $i \le j$, $b^\theta_{i,j}$ is the chance of moving from i to $k \ge j$. For our generalized policy, with cutoffs $(P_{i,j})_{j=1}^N$ after a distraction in memory state i, (37) becomes $\lambda^\theta_{i,j} = F^\theta\left(\log \frac{P_{i,j}}{Q_i}\right) - F^\theta\left(\log \frac{P_{i,j-1}}{Q_i}\right)$. Write as:

$$\lambda^\theta_{i,j} = \begin{cases} a^\theta_{i,j} - a^\theta_{i,j-1} & \text{if } j < i \\ b^\theta_{i,j-1} - b^\theta_{i,j} & \text{if } j > i \end{cases}, \text{ where } a^\theta_{i,0} \equiv 0, b^\theta_{i,N} \equiv 0 \text{ and} \tag{31}$$

$$a^L_{i,j} \equiv \frac{\xi+1}{2\xi}\left(\frac{P_{i,j}}{Q_i}\right)^{\frac{1}{2}(\xi-1)} \; (i > j) \text{ and } b^L_{i,j} \equiv \frac{\xi-1}{2\xi}\left(\frac{Q_i}{P_{i,j}}\right)^{\frac{1}{2}(\xi+1)} \; (i \le j) \tag{32}$$

with $a^H_{i,j}$ and $b^H_{i,j}$ defined by interchanging $\xi - 1$ and $\xi + 1$ in (32). Observe that

$$\frac{da^H_{i,j}}{da^L_{i,j}} = \frac{\partial a^H_{i,j}}{\partial\left(\frac{P_{i,j}}{Q_i}\right)} \Bigg/ \frac{\partial a^L_{i,j}}{\partial\left(\frac{P_{i,j}}{Q_i}\right)} = \frac{P_{i,j}}{Q_i}, \text{ and similarly } \frac{db^H_{i,j}}{db^L_{i,j}} = \frac{P_{i,j}}{Q_i} \tag{33}$$

By (31) choosing the $\lambda^\theta_{i,j}$'s optimally is equivalent to choosing all $a^\theta_{i,j}$ and $b^\theta_{i,j}$ optimally. Now, compute the derivative of payoff $\mu\mathcal{V}^H + \mathcal{V}^L$ in $a^L_{i,j}$, with $a^H_{i,j}$ as a function of $a^L_{i,j}$ via (33). By (31), $a^\theta_{i,j}$ appears in both $\lambda^\theta_{i,j} \equiv a^\theta_{i,j} - a^\theta_{i,j-1}$ and $\lambda^\theta_{i,j+1} \equiv a^\theta_{i,j+1} - a^\theta_{i,j}$ but no other transition chances. So the total derivative of $\mu\mathcal{V}^H + \mathcal{V}^L$ in $a^L_{i,j}$, namely $\mu\frac{da^H_{i,j}}{da^L_{i,j}}\frac{\partial\mathcal{V}^H}{\partial a^H_{i,j}} + \frac{\partial\mathcal{V}^L}{\partial a^L_{i,j}}$, is

$$\begin{aligned} &\mu\frac{da^H_{i,j}}{da^L_{i,j}}\left(\frac{\partial\mathcal{V}^H}{\partial\lambda^H_{i,j}}\frac{\partial\lambda^H_{i,j}}{\partial a^H_{i,j}} + \frac{\partial\mathcal{V}^H}{\partial\lambda^H_{i,j+1}}\frac{\partial\lambda^H_{i,j+1}}{\partial a^H_{i,j}}\right) + \left(\frac{\partial\mathcal{V}^L}{\partial\lambda^L_{i,j}}\frac{\partial\lambda^L_{i,j}}{\partial a^L_{i,j}} + \frac{\partial\mathcal{V}^L}{\partial\lambda^L_{i,j+1}}\frac{\partial\lambda^L_{i,j+1}}{\partial a^L_{i,j}}\right) \\ = \; &\mu\frac{da^H_{i,j}}{da^L_{i,j}}\left(\frac{\partial\mathcal{V}^H}{\partial\lambda^H_{i,j}}(1) + \frac{\partial\mathcal{V}^H}{\partial\lambda^H_{i,j+1}}(-1)\right) + \left(\frac{\partial\mathcal{V}^L}{\partial\lambda^L_{i,j}}(1) + \frac{\partial\mathcal{V}^L}{\partial\lambda^L_{i,j+1}}(-1)\right) \end{aligned}$$

Using the Step 2 expression for $\frac{\partial\mathcal{V}^\theta}{\partial\lambda^\theta_{i,j}}$ along with (33), this is $-\frac{1-\eta}{\eta}$ times

$$\begin{aligned} &\left[\mu\left(\frac{P_{i,j}}{Q_i}\right)\left(\rho^H_i(\nu^H_i - \nu^H_j) - \rho^H_i(\nu^H_i - \nu^H_{j+1})\right) - \left(\rho^L_i(\nu^L_j - \nu^L_i) - \rho^L_i(\nu^L_{j+1} - \nu^L_i)\right)\right] \\ = \; &\rho^L_i\left[P_{i,j}\frac{\mu\rho^H_i}{\rho^L_i Q_i}(\nu^H_{j+1} - \nu^H_j) - (\nu^L_j - \nu^L_{j+1})\right] \end{aligned} \tag{34}$$

Similarly, we have from (31) that $b^\theta_{i,j}$ appears in both $\lambda^\theta_{i,j} \equiv b^\theta_{i,j-1} - b^\theta_{i,j}$ and $\lambda^\theta_{i,j+1} \equiv b^\theta_{i,j} - b^\theta_{i,j+1}$ but no other transition chances. So the payoff derivative in $b^L_{i,j}$, viewing $b^H_{i,j}$ as a function of

$b^L_{i,j}$ via (33), is:

$$\begin{aligned}&\mu\frac{db^H_{i,j}}{db^L_{i,j}}\left(\frac{\partial\mathcal{V}^H}{\partial\lambda^H_{i,j}}\frac{\partial\lambda^H_{i,j}}{\partial b^H_{i,j}}+\frac{\partial\mathcal{V}^H}{\partial\lambda^H_{i,j+1}}\frac{\partial\lambda^H_{i,j+1}}{\partial b^H_{i,j}}\right)+\left(\frac{\partial\mathcal{V}^L}{\partial\lambda^L_{i,j}}\frac{\partial\lambda^L_{i,j}}{\partial b^L_{i,j}}+\frac{\partial\mathcal{V}^L}{\partial\lambda^L_{i,j+1}}\frac{\partial\lambda^L_{i,j+1}}{\partial b^L_{i,j}}\right)\\ = \;&\mu\frac{db^H_{i,j}}{db^L_{i,j}}\left(\frac{\partial\mathcal{V}^H}{\partial\lambda^H_{i,j}}(-1)+\frac{\partial\mathcal{V}^H}{\partial\lambda^H_{i,j+1}}(1)\right)+\left(\frac{\partial\mathcal{V}^L}{\partial\lambda^L_{i,j}}(-1)+\frac{\partial\mathcal{V}^L}{\partial\lambda^L_{i,j-1}}(+1)\right)\end{aligned}$$

Using the Step 2 formula for $\partial\mathcal{V}/\partial\lambda^\theta_{i,j}$ and the second expression in (33), this is $\frac{1-\eta}{\eta}$ times

$$\begin{aligned}&\mu\frac{P_{i,j}}{Q_i}\left(\rho^H_i(\nu^H_j-\nu^H_i)(-1)+\rho^H_i(\nu^H_{j+1}-\nu^H_i)\right)+\left(-\rho^L_i(\nu^L_j-\nu^L_i)+\rho^L_i(\nu^L_{j+1}-\nu^L_i)\right)\\ = \;&\rho^L_i\left[P_{i,j}\frac{\mu\rho^H_i}{\rho^L_i Q_i}(\nu^H_{j+1}-\nu^H_j)-(\nu^L_j-\nu^L_{j+1})\right] \qquad (35)\end{aligned}$$

Step 4 *The Optimal Interval Policy Thresholds*

Lemma B.1 *Given* ***any*** *beliefs $(Q_j)^N_{j=1}$, the optimal $P_{i,j}$ leaves Dug indifferent between post-amnesia values in states j and $j+1$ at "belief" (likelihood ratio) $P_{i,j}(\mu\rho^H_i/Q_i\rho^L_i)$. If (36) below is positive (negative), Dug's payoff is falling (rising) in P_i, j.*

Proof: First consider $P_{i,j}$ with $i>j$. By (32), $P_{i,j}$ appears in $a^\theta_{i,j}$ but no other cumulative transition chances, and (34) gives the total payoff derivative in $a^L_{i,j}$, with $a^H_{i,j}$ viewed as a function of $a^L_{i,j}$. Thus, differentiating $a^L_{i,j}=\frac{\xi+1}{2\xi}\left(\frac{P_{i,j}}{Q_i}\right)^{\frac{1}{2}(\xi-1)}$ (from (32)) in $P_{i,j}$ and combining with (34), the payoff derivative $\partial\mathcal{V}/\partial P_{i,j}$ is:

$$\frac{\partial\mathcal{V}}{\partial a^L_{i,j}}\frac{\partial a^L_{i,j}}{\partial P_{i,j}}=-\frac{1-\eta}{\eta}\rho^L_i\left[P_{i,j}\frac{\mu\rho^H_i}{\rho^L_i Q_i}(\nu^H_{j+1}-\nu^H_j)-(\nu^L_j-\nu^L_{j+1})\right]\cdot\frac{\xi+1}{2\xi}\frac{\xi-1}{2P_{i,j}}\left(\frac{P_{i,j}}{Q_i}\right)^{\frac{\xi-1}{2}}$$

And for $\imath\le j$, where (by (32)) $P_{i,j}$ appears in $b^\theta_{i,j}\equiv\frac{\xi-1}{2}\left(\frac{Q_i}{P_{i,j}}\right)^{\frac{\xi+1}{2}}$ but no other cumulative transition chances, differentiate $b^L_{i,j}$ and combine with (35) to obtain $\partial\mathcal{V}/\partial P_{i,j}$:

$$\frac{\partial\mathcal{V}}{\partial b^L_{i,j}}\frac{\partial b^L_{i,j}}{\partial P_{i,j}}=\frac{1-\eta}{\eta}\rho^L_i\left[P_{i,j}\frac{\mu\rho^H_i}{\rho^L_i Q_i}(\nu^H_{j+1}-\nu^H_j)-(\nu^L_j-\nu^L_{j+1})\right]\left(-\frac{\xi-1}{2\xi}\frac{\xi+1}{2P_{i,j}}\left(\frac{Q_i}{P_{i,j}}\right)^{\frac{\xi+1}{2}}\right)$$

Together, these two expressions imply that for all i and j, $\partial\mathcal{V}/P_{i,j}$ has the opposite sign to:

$$P_{i,j}\frac{\mu\rho^H_i}{\rho^L_i Q_i}(\nu^H_{j+1}-\nu^H_j)-(\nu^L_j-\nu^L_{j+1}) \qquad (36)$$

This proves the final assertion in Lemma B.1, and establishes that (36) must vanish for *interior* optimal cutoffs $P_{i,j}$. We now prove that it optimally vanishes also in corner solutions. Specifically, we rule out "lower corner solutions" where $\partial\mathcal{V}/\partial P_{i,j} < 0$ but $P_{i,j}$ is set to its minimum possible value, namely $P_{i,j-1}$. (A symmetric argument rules out upper corner solutions, where $\partial\mathcal{V}/\partial P_{i,j} > 0$ with $P_{i,j}$ set to its maximum). Toward a contradiction, assume an optimal memory where for some i and j, $P_{i,j} = P_{i,j-1}$ and $\partial\mathcal{V}/\partial P_{i,j} < 0$ (so (36) is positive). Given i, let j^* be the smallest such j. We know $j^* \geq 2$, since (36) cannot be strictly positive at $P_{i,1} = P_{i,0} \equiv 0$. So by construction, P_{i,j^*-1} is interior, thus (by optimality) Dug is indifferent between expected post-amnesia payoffs in $j^* - 1$ and j^* at "belief" $P_{i,j^*-1}\frac{\mu\rho_i^H}{\rho_i^L Q_i}$. But by construction ((36) is positive for j^* and $P_{i,j^*} = P_{i,j^*-1}$), he strictly prefers $j^* + 1$ to j^* at this belief, and thus also to $j^* - 1$. But this contradicts optimality: $P_{i,j^*} = P_{i,j^*-1}$ implies that Dug never transitions $i \to j^*$ between distractions, so P_{i,j^*-1} is the effective threshold between $j^* - 1$ and $j^* + 1$, and since it is interior by construction, optimality requires indifference. This proves the first assertion in Lemma B.1. □

Step 5 *Bayesian Beliefs are Optimal. (Proof of Lemma 1)*

We now consider the optimal post-amnesia beliefs Q_i. Firstly they impact the payoff via the $a_{i,j}^\theta$ and $b_{i,j}^\theta$ terms, but by Step 4, optimal cutoffs ensure that this effect vanishes. So given optimal cutoffs and Step 1, the payoff derivative in Q_i has the same sign as $\mu\rho_i^H/\rho_i^L - Q_i$, which is positive at a below-Bayesian belief (Dug should raise Q_i) and negative at an above-Bayesian belief (Dug should lower Q_i). Thus Bayesian beliefs are optimal. □

Step 6 *Indifference as a FOC (Proof of Proposition 1).*

Substituting $Q_i = \mu\rho_i^H/\rho_i^L$ from Step 5 into equation (36), *every* $P_{i,j}$ solves the same indifference condition, namely (36) vanishes. Setting $P_{i,j} = P_j$ and applying Bayes consistency, this rearranges to (13), completing the proof. □

B.1 Proof of Corollary 1

Toward a contradiction, assume an optimal corner solution ($P_i = P_{i+1}$ for some i). Then Dug never uses memory state i, so consider instead adding a new memory state "0", with threshold $P_0 < P_1$ and Bayesian beliefs Q_0; by Step 1 of Definition 1 below, this implies $Q_0/P_0 = \underline{x}$, where $\underline{x} \in (0, 1)$ is the root of (38). We show that for any P_0 sufficiently small (where $P_0 = 0$ means the state is unused), Dug gains by raising P_0. To this end, consider the limit $P_0 \to 0$, where $Q_0 = P_0\underline{x} \to 0$. By (10), $w_0^L \equiv \pi F^L\left(\frac{\pi}{\Pi Q_0}\right) \to \pi$. And by (37),

$$\lambda_{0,1}^L \to \frac{\xi-1}{2\xi}\underline{x}^{\frac{\xi+1}{2}} > 0 \text{ and } \lambda_{0,j}^L = \frac{\xi-1}{2\xi}\underline{x}^{\frac{\xi+1}{2}}\left(\left(\frac{P_0}{P_{j-1}}\right)^{\frac{\xi+1}{2}} - \left(\frac{P_0}{P_j}\right)^{\frac{\xi+1}{2}}\right) \to 0 \quad \forall j \geq 2$$

Thus (12) becomes

$$\begin{aligned} &v_0^L \to \eta\pi + (1-\eta)\frac{\xi-1}{2\xi}\underline{x}^{\frac{1}{2}(\xi+1)}v_1^L + (1-\eta)\left(1 - \frac{\xi-1}{2\xi}\underline{x}^{\frac{1}{2}(\xi+1)}\right)v_0^L \\ &\Rightarrow \lim_{P_0\to 0}(1-\eta)\frac{\xi-1}{2\xi}\underline{x}^{\frac{1}{2}(\xi+1)}(v_0^L - v_1^L) = \eta(\pi - v_0^L) \end{aligned}$$

Since optimally $\nu_1^L < \pi$ ($\nu_0^L = \nu_1^L = \pi$ means Dug ultimately chooses $\mathcal{L}$ with probability 1, learning nothing), this implies that $\nu_0^L - \nu_1^L$ remains boundedly positive, and so $P_1(\nu_1^H - \nu_0^H) - (\nu_0^L - \nu_1^L)$ remains boundedly negative as $P_0 \to 0$. That is, Dug strictly prefers memory state 0 to 1 at threshold P_0, and so by Proposition (1) should raise P_0. □

C Post-Amnesia Beliefs: Proof of Proposition 2

We present a recursive algorithm which yields unique post-amnesia beliefs $\mathbf{q} = (q_1, q_2, \ldots, q_N)$ for any interval policy $\mathbf{p} = (p_1, \ldots, p_{N-1})$ and prove that it satisfies (9) *iff* it obeys our recursion. We use $Q_n \equiv q_n/(1-q_n)$ and $P_n \equiv p_n/(1-p_n)$; and so, by (6) and (5):

$$\lambda_{i,j}^H = \begin{cases} \frac{\xi+1}{2\xi}\left(\left(\frac{Q_i}{P_{j-1}}\right)^{\frac{\xi-1}{2}} - \left(\frac{Q_i}{P_j}\right)^{\frac{\xi-1}{2}}\right) & \text{if } i < j \\ \frac{\xi-1}{2\xi}\left(\left(\frac{P_j}{Q_i}\right)^{\frac{\xi+1}{2}} - \left(\frac{P_{j-1}}{Q_i}\right)^{\frac{\xi+1}{2}}\right) & \text{if } i > j \end{cases} \tag{37}$$

And $\lambda_{i,j}^L$ just swaps $\xi - 1$ and $\xi + 1$, while $\lambda_{i,i}^\theta = 1 - \sum_{j\neq i}\lambda_{i,j}^\theta$.

Fis an interval policy. Let n_0 denote the *initial state*; namely the index of the memory state with $\mu \in [P_{n_0-1}, P_{n_0}]$. Define $\gamma_n \equiv P_n/P_{n+1} \leq 1$ for $n < N-1$ with $\gamma_0 = 0$ and $\gamma_{N-1} = 1$, and the functions $f : \mathbb{R} \mapsto \mathbb{R}, g : \mathbb{R}^2 \mapsto \mathbb{R}$, and $\beta : \mathbb{R} \mapsto \mathbb{R}$ as follows:

$$f(R) = \frac{\eta}{1-\eta}\left(R - \frac{\xi-1}{\xi+1}\right) + \frac{2}{\xi+1}R^{\frac{\xi+1}{2}} \tag{38}$$

$$g(R,\gamma) = \frac{\frac{2\eta\xi}{(1-\eta)(\xi+1)}(\beta(\gamma) - R) + \left(\frac{\gamma}{R}\right)^{\frac{\xi-1}{2})}\left(\beta(\gamma) - \frac{\xi-1}{\xi+1}\gamma\right) - R^{\frac{\xi+1}{2}}\left(1 - \frac{\xi-1}{\xi+1}\beta(\gamma)\right)}{\left(\frac{(\beta(\gamma))^2}{\gamma} - 1\right)\left(\gamma^{\frac{1-\xi}{2}} - 1\right)} \tag{39}$$

$$\beta(\gamma) = \left(\frac{\xi-1}{\xi+1}\right)\left(1 - \gamma^{\frac{\xi+1}{2}}\right) / \left(1 - \gamma^{\frac{\xi-1}{2}}\right) \tag{40}$$

Definition 1 *The Outside-In Recursion constructs $(Q_n)_{n=1}^N$ recursively as follows:*

1. *If $n_0 \neq 1$, then Q_1 solves $f(Q_1/P_1) = 0$. If $n_0 \neq N$, Q_N solves $f(P_{N-1}/Q_N) = 0$.*

2. *Given Q_k and $\phi_k \equiv \rho_k^L/\rho_1^L$ for $k \leq n-1$, Q_n and ϕ_n $\forall n < n_0$ solve*

$$\frac{f\left(\frac{Q_n}{P_n}\right)}{g\left(\frac{Q_n}{P_n}, \gamma_{n-1}\right)} = \frac{-\sum\limits_{k\leq n-1} \phi_k f\left(\frac{Q_k}{P_n}\right)}{\sum\limits_{k\leq n-1} \phi_k \left(\frac{Q_k}{P_n}\right)^{\frac{\xi+1}{2}}} \quad \text{and} \quad \phi_n = \frac{-\sum\limits_{k\leq n-1} \phi_k f\left(\frac{Q_k}{P_n}\right)}{f(\frac{Q_n}{P_n})} \tag{41}$$

3. *Given Q_k and $\Phi_k \equiv \rho_k^H/\rho_N^H$ for $k \geq n+1$, Q_n and Φ_n $\forall n > n_0$ solve*

$$\frac{f\left(\frac{P_{n-1}}{Q_n}\right)}{g\left(\frac{P_{n-1}}{Q_n}, \gamma_{n-1}\right)} = \frac{-\sum\limits_{k\geq n+1} \Phi_k f\left(\frac{P_{n-1}}{Q_k}\right)}{\sum\limits_{k\geq n+1} \Phi_k \left(\frac{P_{n-1}}{Q_k}\right)^{\frac{1}{2}(\xi+1)}} \quad \text{and} \quad \Phi_n = \frac{-\sum\limits_{k\geq n+1} \Phi_k f\left(\frac{P_{n-1}}{Q_k}\right)}{f(\frac{P_{n-1}}{Q_n})} \tag{42}$$

4. *In the initial state, Q_{n_0} is the root of the following equation, where the coefficients $(X_i)_{i=1}^4$ (given by (44)) depend on η, ξ, and the average beliefs $\overline{Q}_{n_0-1}$ in states $k \leq n_0-1$ and $\underline{Q}_{n_0+1}$ in states $k \geq n_0+1$, given $\theta = L$:*

$$\left[X_1 - \frac{\left(\underline{Q}_{n_0+1} - \mu\right)}{\mu - \overline{Q}_{n_0-1}} X_3\right] \left(\frac{Q_{n_0}}{P_{n_0}}\right)^{\frac{1}{2}(\xi+1)} + \left[X_2 - \frac{\left(\underline{Q}_{n_0+1} - \mu\right)}{\mu - \overline{Q}_{n_0-1}} X_4\right] \left(\frac{P_{n_0-1}}{Q_{n_0}}\right)^{\frac{1}{2}(\xi-1)}$$
$$+ \frac{\eta}{1-\eta} \frac{\xi-1}{2} \frac{\mu - Q_{n_0}}{\mu - \overline{Q}_{n_0-1}} \left(X_2 X_3 - X_1 X_4\right) \tag{43}$$

where

$$\begin{array}{ll} X_1 = \left(\frac{\underline{Q}_{n_0+1}}{P_{i_0}} - \frac{\xi+1}{\xi-1}\right) \gamma_{n_0-1}^{\frac{1}{2}(\xi-1)} & X_2 = \frac{\xi+1}{\xi-1} \frac{\underline{Q}_{n_0+1}}{P_{n_0}} - 1 \\ X_3 = \frac{\xi+1}{\xi-1} - \frac{\overline{Q}_{n_0-1}}{P_{n_0-1}} & X_4 = \gamma_{n_0-1}^{\frac{1}{2}(\xi+1)} \left(1 - \frac{\xi+1}{\xi-1} \frac{\overline{Q}_{n_0-1}}{P_{i_0-1}}\right) \end{array} \tag{44}$$

5. *Appendix C.5 also solves for $\rho_1^L, \rho_{n_0}^L$, and ρ_N^L in terms of coefficients X_5 and X_6 in (45), then remaining ρ_k^θ values follow using ϕ_k and Φ_k (Steps 2 and 3) and Bayes consistency:*

$$\rho_{n_0}^L = \frac{1}{1 + X_5 + X_6}, \quad \rho_1^L = \frac{X_5 \rho_{n_0}^L}{\sum\limits_{j\leq n_0-1} \phi_j}, \quad \rho_N^L = \frac{X_6 \rho_{n_0}^L}{\sum\limits_{j\geq i_0+1} \Phi_j \frac{Q_N}{Q_j}}, \text{ where} \tag{45}$$

$$X_5 = \frac{X_1 \left(\frac{Q_{n_0}}{P_{n_0}}\right)^{\frac{1}{2}(\xi+1)} + X_2 \left(\frac{P_{n_0-1}}{Q_{n_0}}\right)^{\frac{1}{2}(\xi-1)}}{\frac{\eta}{1-\eta} \frac{\xi-1}{2} \left(X_2 X_3 - X_1 X_4\right)}, X_6 = \frac{X_3 \left(\frac{Q_{n_0}}{P_{n_0}}\right)^{\frac{1}{2}(\xi+1)} + X_4 \left(\frac{P_{n_0-1}}{Q_{n_0}}\right)^{\frac{1}{2}(\xi-1)}}{\frac{\eta}{1-\eta} \frac{\xi-1}{2} \left(X_2 X_3 - X_1 X_4\right)}$$

C.1 Post-Amnesia Beliefs Obey the Recursion for $n \neq n_0$

Lemma C.1 *Any fixed point* $(Q_n)_{n=1}^N$ *of* (9) *satisfies the Outside-In Recursion* $\forall n \neq n_0$.

Proof: We prove the $n < n_0$ case. Symmetric steps establish the $n > n_0$ case. The expression $f(Q_1/P_1) = 0$ in Part 1 is precisely the second expression in (41) at $n = 1$, so we prove that this expression holds $\forall 1 \leq n < n_0$, and that the first expression in (41) holds $\forall 2 \leq n < n_0$. Throughout, we exploit the fact that by (9), ρ^θ is the steady-state distribution of a Markov process that jumps to n_0 with chance η and otherwise transitions according to $\lambda^\theta_{i,j}$, and so probability masses entering and leaving any block of states must balance.

Step 1 $(Q_n)_{n=1}^N$ *is a fixed point of* (9) *iff it obeys the second equation in* (41) $\forall 1 \leq n < n_0$.

Using the fact that the mass leaving states $\{1, 2, ..., n\}$ equals the mass entering these states and taking ratios (H to L):

$$\frac{\sum_{k\leq n} \rho^H_k \left(\eta + (1-\eta)\sum_{j\geq n+1} \lambda^H_{k,j}\right)}{\sum_{k\leq n} \rho^L_k \left(\eta + (1-\eta)\sum_{j\geq n+1} \lambda^L_{k,j}\right)} = \frac{\sum_{j\geq n+1} \rho^H_j \sum_{k\leq n}(1-\eta)\lambda^H_{j,k}}{\sum_{j\geq n+1} \rho^L_j \sum_{k\leq n}(1-\eta)\lambda^L_{j,k}} \tag{46}$$

We have from (37) and Bayes consistency ($Q_n = \mu\rho^H_n/\rho^L_n$) that for *any* $j \geq n+1$,

$$\frac{\rho^H_j}{\rho^L_j}\frac{\sum_{k\leq n}\lambda^H_{j,k}}{\sum_{k\leq n}\lambda^L_{j,k}} = \frac{Q_j}{\mu}\frac{\frac{\xi-1}{2\xi}\left(\frac{P_n}{Q_j}\right)^{\frac{\xi+1}{2}}}{\frac{\xi+1}{2\xi}\left(\frac{P_n}{Q_j}\right)^{\frac{\xi-1}{2}}} \equiv \frac{\xi-1}{\xi+1}\frac{P_n}{\mu}$$

So the RHS of (46) equals $\frac{\xi-1}{\xi+1}\frac{P_n}{\mu}$. Also simplifying the LHS using $\mu\rho^H_k \equiv \rho^L_k Q_k$, then dividing LHS numerator and denominator by ρ^L_1, and LHS and RHS by P_n/μ, (46) becomes

$$\frac{\sum_{k\leq n-1}\phi_k\frac{Q_k}{P_n}\left(\frac{\eta}{1-\eta} + \sum_{j\geq n+1}\lambda^H_{k,j}\right) + \phi_n\frac{Q_n}{P_n}\left(\frac{\eta}{1-\eta} + \sum_{j\geq n+1}\lambda^H_{n,j}\right)}{\sum_{k\leq n-1}\phi_k\left(\frac{\eta}{1-\eta} + \sum_{j\geq n+1}\lambda^L_{k,j}\right) + \phi_n\left(\frac{\eta}{1-\eta} + \sum_{j\geq n+1}\lambda^L_{n,j}\right)} = \frac{\xi-1}{\xi+1} \tag{47}$$

By (37) for $k \leq n$ we have $\sum_{j\geq n+1}\lambda^H_{k,j} = \frac{\xi+1}{2\xi}\left(\frac{Q_k}{P_n}\right)^{\frac{\xi-1}{2}}$, and just swap $\xi - 1$ with $\xi + 1$ in state L. Plugging this into (47) and solving for ϕ_n yields the second expression in (41). $\square$.

Step 2 $(Q_n)_{n=1}^N$ *is a fixed point of* (9) *iff it obeys the first equation in* (41) $\forall 2 \leq n < n_0$.

Using the fact that probability mass leaving state n equals probability mass entering state n, grouping the expression to have all ρ^θ_k terms with $k \leq n$ on the LHS, and taking ratios (H to L) yields:

$$\frac{\rho_n^H\left(1-(1-\eta)\lambda_{n,n}^H\right)-\sum_{k\le n-1}\rho_k^H(1-\eta)\lambda_{k,n}^H}{\rho_n^L\left(1-(1-\eta)\lambda_{n,n}^L\right)-\sum_{k\le n-1}\rho_k^L(1-\eta)\lambda_{k,n}^L}=\frac{\sum_{j\ge n+1}\rho_j^H(1-\eta)\lambda_{j,n}^H}{\sum_{j\ge n+1}\rho_j^L(1-\eta)\lambda_{j,n}^L}$$

The RHS equals $P_n\cdot\beta(\gamma_{n-1})/\mu$ (with β defined in (40)), since by (37) for $\forall j\ge n+1$,

$$\mu\frac{\rho_j^H}{\rho_j^L}\frac{\lambda_{j,n}^H}{\lambda_{j,n}^L}\equiv Q_j\frac{\frac{\xi-1}{2\xi}\left(\left(\frac{P_n}{Q_j}\right)^{\frac{\xi+1}{2}}-\left(\frac{P_{n-1}}{Q_j}\right)^{\frac{\xi+1}{2}}\right)}{\frac{\xi+1}{2\xi}\left(\left(\frac{P_n}{Q_j}\right)^{\frac{\xi-1}{2}}-\left(\frac{P_{n-1}}{Q_j}\right)^{\frac{\xi-1}{2}}\right)}=P_n\frac{\xi-1}{\xi+1}\frac{1-\gamma_{n-1}^{\frac{\xi+1}{2}}}{1-\gamma_{n-1}^{\frac{\xi-1}{2}}}\equiv P_n\cdot\beta(\gamma_{n-1})$$

Using this and simplifying the LHS using first $\mu\rho_j^H\equiv\rho_j^L Q_j$, then dividing numerator and denominator by $(1-\eta)\rho_1^L$, then dividing LHS and RHS by P_n/μ, we obtain:

$$\frac{\phi_n\frac{Q_n}{P_n}\left(\frac{1}{1-\eta}-\lambda_{n,n}^H\right)-\sum_{k\le n-1}\phi_k\frac{Q_k}{P_n}\lambda_{k,n}^H}{\phi_n\left(\frac{1}{1-\eta}-\lambda_{n,n}^L\right)-\sum_{k\le n-1}\phi_k\lambda_{k,n}^L}=\beta_{n-1}\Rightarrow\phi_n=\frac{\sum_{k\le n-1}\phi_k\left(\lambda_{k,n}^L\beta_{n-1}-\frac{Q_k}{P_n}\lambda_{k,n}^H\right)}{\left(\frac{1}{1-\eta}-\lambda_{n,n}^L\right)\beta_{n-1}-\frac{Q_n}{P_n}\left(\frac{1}{1-\eta}-\lambda_{n,n}^H\right)}$$

Using transition chances from (37), simplify using (39) as $\phi_n=\sum\limits_{k\le n-1}\phi_k\left(\frac{Q_k}{P_n}\right)^{\frac{\xi+1}{2}}/g(\frac{Q_n}{P_n},\gamma_{n-1})$. Equate the RHS of this expression with the RHS of the second expression in (41) (verified in Step 1), and solve for Q_n/P_n to obtain the first expression in(41). □

C.2 Existence, Uniqueness, and Bounds for $n\neq n_0$

We now prove (for $n\neq n_0$) that there is a unique solution Q_n to our Recursion, with $Q_n\in[P_{n-1},P_n]$. First, we establish key properties of functions β, f, and g from (40),(38),(39):

Lemma C.2 *If $\gamma<1$, then $\beta(\gamma)>\frac{\xi-1}{\xi+1}$, $\beta(\gamma)>\sqrt{\gamma}$, and $\beta(\gamma)<1$. Thus, f as defined in* (38) *is strictly increasing in R, and g defined in* (39) *is strictly decreasing in R.*

Proof: The second sentence follows by immediate inspection of (38) and (39) if the three inequalities for β in the first sentence hold. The first β inequality follows directly from the definition of β and $\gamma<1$. For the second, let $x\equiv\sqrt{\gamma}$, so that

$$\frac{\beta}{\sqrt{\gamma}}\equiv\frac{\beta}{x}\equiv\frac{\xi-1}{\xi+1}\frac{1-x^{\xi+1}}{x-x^{\xi}}\tag{48}$$

We first claim that the expression in (48) decreases in x. Indeed, its derivative is negative iff:

$$-(\xi+1)x^{\xi} - \frac{1-x^{\xi+1}}{x-x^{\xi}}\left(1-\xi x^{\xi-1}\right) < 0 \Leftrightarrow x^{2\xi} + x^{\xi-1}\xi(1-x^2) - 1 < 0 \tag{49}$$

To show that the final expression in (49) is negative $\forall x < 1$, it suffices to show that it's increasing in x, since it vanishes at $x = 1$. Differentiating, its derivative has the same sign as $\xi - 1 - (\xi+1)x^2 + 2x^{\xi+1}$; this is positive, as desired, since it is itself decreasing in $x < 1$ by an immediate calculation, thus at least its zero value at $x = 1$. So (49) holds, thus β/x decreases in x and so is at least its value in the limit as $x \to 1$, which by l'hopital's rule is 1. Thus $\beta/\sqrt{\gamma} \geq 1$. Finally for $\beta < 1$: multiplying (48) by x and differentiating in x yields

$$\frac{d\beta}{dx} = \frac{d}{dx}\frac{1-x^{\xi+1}}{1-x^{\xi-1}} = \frac{-(\xi+1)x^{\xi} + \frac{1-x^{\xi+1}}{1-x^{\xi-1}}(\xi-1)x^{\xi-2}}{1-x^{\xi-1}} = \frac{(\xi+1)\,x^{\xi-2}}{1-x^{\xi-1}}\left(\beta - x^2\right)$$

This is positive by the second claim, $\beta > x$, given that $x < 1$ (so $x > x^2$). Thus β is increasing in x, and thus is at most its limit as $x \to 1$, which by l'hopital's rule is 1. □

Lemma C.3 *$\forall n \neq n_0$, the Outside-In Recursion has a unique solution, with $Q_n \in [P_{n-1}, P_n]$.*

PROOF: We prove the $n < n_0$ case by induction. Symmetric logic applies to $n > n_0$.

Step 1 *Proof of Lemma for $n = 1$*

By Definition 1 Part 1, Q_1/P_1 is the root $\underline{x}$ of f: By Lemma C.2 f is increasing in R, and by immediate inspection of (38), f is negative at $R = 0$ and positive at $R = 1$. Thus it has a unique root $\underline{x}$ between 0 and 1, and so $0 < Q_1/P_1 < 1$. □.

Now assume the Lemma is true for $k < n$. We prove it for $k = n$ via the following steps.

Step 2 *$g(1,\gamma) < 0$ and $g(\underline{x},\gamma) > 0$ for any $\gamma \leq 1$*

For the first claim, using (39), the numerator of $g(R,\gamma)$ evaluated at $R = 1$ is

$$\begin{aligned} &\frac{2\eta\xi}{(1-\eta)(\xi+1)}(\beta-1) + \gamma^{\frac{\xi-1}{2}}\left(\beta - \frac{\xi-1}{\xi+1}\gamma\right) - \left(1 - \frac{\xi-1}{\xi+1}\beta\right) \\ < \quad &\gamma^{\frac{1}{2}(\xi-1)}\left(1 - \frac{\xi-1}{\xi+1}\gamma\right) - \left(1 - \frac{\xi-1}{\xi+1}\right) \quad \text{by } \beta < 1 \text{ (Lemma C.2)} \end{aligned}$$

Collecting terms, this is precisely $-(1-\gamma^{\frac{1}{2}(\xi-1)})\cdot(1-\beta)$, at most 0 by $\beta \leq 1$. For the second claim, clearly the denominator of $g(R,\gamma)$ in (39) is positive, so consider the numerator. By inspection it is decreasing in β, so by Lemma C.2 it is at least its value at β lower bound $\frac{\xi-1}{\xi+1}$, which simplifies to $-f(\underline{x})+\left(\frac{\gamma}{\underline{x}}\right)^{\frac{\xi-1}{2}}\frac{\xi-1}{\xi+1}(1-\gamma)$. This is positive by $f(\underline{x})=0$. $\square$.

Step 3 *The RHS of the first expression in* (41) *exceeds* $\frac{2}{\xi+1}\left(1-\gamma_{n-1}^{\frac{\xi+1}{2}}\right)\left(\frac{1}{\gamma_{n-1}}\right)^{\frac{\xi+1}{2}}$

Rewrite its numerator as:

$$-\sum_{k\leq n-1}\phi_k f\left(\frac{Q_k}{P_n}\right)=-\sum_{k\leq n-1}\phi_k\left[f\left(\gamma_{n-1}\frac{Q_k}{P_{n-1}}\right)\right] \tag{50}$$

But by (38), for any x and γ we have

$$\begin{aligned} f(x)-f(\gamma x) &= \frac{\eta}{1-\eta}(1-\gamma)x+\frac{2}{\xi+1}x^{\frac{\xi+1}{2}}\left(1-\gamma^{\frac{\xi+1}{2}}\right) \\ \Rightarrow -f(\gamma x) &> -f(x)+\frac{2}{\xi+1}x^{\frac{\xi+1}{2}}\left(1-\gamma^{\frac{\xi+1}{2}}\right) \end{aligned} \tag{51}$$

Using (51) at $\gamma=\gamma_{n-1}$ and $x=Q_k/P_{n-1}$ to replace the square bracketed term in (50), we obtain the following lower bound on the RHS of (50):

$$-\sum_{k\leq n-1}\phi_k f\left(\frac{Q_k}{P_{n-1}}\right)+\frac{2}{\xi+1}(1-\gamma_{i-1}^{\frac{\xi+1}{2}})\sum_{k\leq n-1}\phi_k\left(\frac{Q_k}{P_{n-1}}\right)$$

But the first term vanishes using the second expression in (41) for ϕ_{n-1}, so this is precisely $\frac{2}{\xi+1}\left(1-\gamma_{n-1}^{\frac{\xi+1}{2}}\right)\left(\frac{1}{\gamma_{n-1}}\right)^{\frac{\xi+1}{2}}$ times the RHS denominator of the first expression in (41). $\square$.

Step 4 *At* $\frac{Q_n}{P_n}=\gamma_{n-1}$*, the LHS of the first* (41) *equation is smaller than the Step 3 expression.*

Evaluating (38) and (39) at $R_n=\gamma_{n-1}$, this LHS expression $f(\gamma_{n-1})/g(\gamma_{n-1},\gamma_{n-1})$ is

$$\frac{\gamma_{n-1}^{-\frac{\xi+1}{2}}\left(1-\gamma_{n-1}^{\frac{\xi-1}{2}}\right)\left(\beta_{n-1}^2-\gamma_{n-1}\right)\left(\frac{\eta}{1-\eta}\left(\gamma_{n-1}-\frac{\xi-1}{\xi+1}\right)+\frac{2}{\xi+1}\gamma_{n-1}^{\frac{\xi+1}{2}}\right)}{\frac{2\eta\xi}{(1-\eta)(\xi+1)}\left(\beta_{n-1}-\gamma_{n-1}\right)+\left(\beta_{n-1}-\frac{\xi-1}{\xi+1}\gamma_{n-1}\right)-\gamma_{n-1}^{\frac{\xi+1}{2}}\left(1-\frac{\xi-1}{\xi+1}\beta_{n-1}\right)} \tag{52}$$

Collecting terms, the denominator of (52) rearranges to

$$\begin{aligned} &\frac{2\xi}{(1-\eta)(\xi+1)}\left(\beta_{n-1}-\gamma_{n-1}\right)-\left(1-\gamma_{n-1}^{\frac{\xi-1}{2}}\right)\left(\beta_{n-1}^2-\gamma_{n-1}\right) \\ \geq\ &\left(\frac{2\xi}{(1-\eta)(\xi+1)}-(1-\gamma_{n-1}^{\frac{\xi-1}{2}})\right)(\beta_{n-1}^2-\gamma_{n-1}) \text{ by } \beta_{n-1}\leq 1 \end{aligned}$$

This is positive by Lemma C.2. Using this to replace the denominator in (52), then dividing numerator and denominator by $\beta_{n-1}^2 - \gamma_{n-1}$, we obtain the following upper bound on (52):

$$\left(\frac{1}{\gamma_{n-1}}\right)^{\frac{\xi+1}{2}} \left(1-\gamma_{n-1}^{\frac{\xi-1}{2}}\right) \left[\frac{\frac{\eta}{1-\eta}\left(\gamma_{n-1}-\frac{\xi-1}{\xi+1}\right)+\frac{2}{\xi+1}\gamma_{n-1}^{\frac{\xi+1}{2}}}{\frac{2\xi}{(1-\eta)(\xi+1)}-1+\gamma_{n-1}^{\frac{\xi-1}{2}}}\right]$$

Comparing to the Step 3 expression, it suffices to show that the square bracketed term is at most $\frac{2}{\xi+1}$. This holds since the numerator multiplied by $\frac{\xi+1}{2}$ is at most $\frac{\eta}{1-\eta}+\gamma_{n-1}^{\frac{\xi+1}{2}}$ by $\gamma_{n-1} \leq 1$, while the denominator is at least this large by $\xi > 1$. □

Step 5 *Completing the Proof*

We prove that the first equation in (41) has a unique solution $R_n \equiv Q_n/P_n \in [\gamma_{n-1}, 1]$. By Step 3, the RHS of (41) is positive, so the LHS, namely $f(R_n)/g(R_n, \gamma_{n-1})$ must be too. By Step 2 and since f is increasing by Lemma C.2, this is impossible if $R_n < \underline{x}$, where $f < 0 < g$. Also by Step 2, since g is decreasing by Lemma C.2, g downcrosses at some $\overline{R} \in (\underline{x}, 1)$, so a solution with $R_n > \overline{R}$ is impossible (here $f > 0 >$g). So any solution lies in $[\underline{x}, \overline{R}]$. Here the LHS of (41) is increasing (f is increasing, g decreasing, and both positive), so there is at most one solution. A solution exists by the intermediate value theorem, and lies in $[\gamma_{n-1}, 1]$, since the LHS is too small at $R_n = \gamma_{n-1}$ by Steps 3 and 4, and too large (∞) at $R_n = \overline{R}$. □.

C.3 Post-Amnesia Beliefs Obey the Recursion for $n = n_0$

Define $\overline{Q}_{n_0-1}$ and $\underline{Q}_{n_0+1}$ as (resp.) the average beliefs in states below, above n_0 in $\theta = L$:

$$\overline{Q}_{n_0-1} \equiv \frac{\sum_{k\leq n_0-1}\rho_k^L Q_k}{\sum_{k\leq n_0-1}\rho_k^L} = \frac{\sum_{k\leq n_0-1}\phi_k Q_k}{\sum_{k\leq n_0-1}\phi_k},\ \underline{Q}_{n_0+1} \equiv \frac{\sum_{k\geq n_0+1}\rho_k^L Q_k}{\sum_{k\geq i_0+1}\rho_k^L} = \frac{\sum_{k\geq n_0+1}\Phi_k}{\sum_{k\geq i_0+1}\frac{\Phi_k}{Q_k}}$$

Step 1 *Deriving Key Equations*

Having derived beliefs $Q_n\ \forall n \neq n_0$, and ratios $\phi_n \equiv \rho_n^L/\rho_1^L$ ($\forall n < n_0$) and $\Phi_n \equiv \rho_n^H/\rho_N^H$ ($\forall n > n_0$), we now derive the 3 equations that identify the remaining 3 key variables: Q_{n_0}, $X_5 \equiv \left(\sum_{k\leq n_0-1}\rho_k^L\right)/\rho_{n_0}^L$, and $X_6 \equiv \left(\sum_{k\geq n_0+1}\rho_k^L\right)/\rho_{n_0}^L$. First define the coefficients:

$$\widetilde{B} = \frac{\sum\limits_{k\leq n_0-1}\rho_k^L\left(\frac{Q_k}{P_{n_0-1}}\right)^{\frac{\xi+1}{2}}}{\sum\limits_{k\leq n_0-1}\rho_k^L} \text{ and } \widetilde{b} = \frac{\sum\limits_{k\geq n_0+1}\rho_k^L\left(\frac{P_{n_0}}{Q_k}\right)^{\frac{\xi-1}{2}}}{\sum\limits_{k\geq i_0+1}\rho_k^L} \tag{53}$$

Equating the probabilities of leaving and entering state n_0, we have $\rho^\theta_{n_0}(1-\lambda^\theta_{n_0,n_0}) = \sum_{j\neq n_0}\rho^\theta_j\left(\frac{\eta}{1-\eta}+\lambda^\theta_{j,n_0}\right)$. When $\theta = L$, using (37) and multiplying by $\frac{2\xi}{\xi-1}/\rho^L_{n_0}$, this is

$$\begin{aligned}
&\left(\frac{Q_{n_0}}{P_{n_0}}\right)^{\frac{1}{2}(\xi+1)} + \frac{\xi+1}{\xi-1}\left(\frac{P_{n_0-1}}{Q_{n_0}}\right)^{\frac{1}{2}(\xi-1)} \\
= \;& X_5\left(\frac{\eta}{1-\eta}\frac{2\xi}{\xi-1} + \left(1-\gamma_{n_0-1}^{\frac{\xi+1}{2}}\right)\widetilde{B}\right) + X_6\left(\frac{\eta}{1-\eta}\frac{2\xi}{\xi-1} + \frac{\xi+1}{\xi-1}\left(1-\gamma_{n_0-1}^{\frac{1}{2}(\xi-1)}\right)\widetilde{b}\right)
\end{aligned} \tag{54}$$

And in state $\theta = H$, swapping $\xi+1$ with $\xi-1$ and replacing ρ^L_k by $\rho^H_k = \rho^L_k Q_k/\mu$, and multiplying through by $\frac{2\xi}{\xi+1}\frac{Q_{n_0}}{P_{n_0}}$, we obtain the following analog to equation (54):

$$\begin{aligned}
\left(\frac{Q_{n_0}}{P_{n_0}}\right)^{\frac{\xi+1}{2}} + \frac{\xi-1}{\xi+1}\gamma_{n_0-1}\left(\frac{P_{n_0-1}}{Q_{n_0}}\right)^{\frac{\xi-1}{2}} &= X_5\left(\frac{\eta}{1-\eta}\frac{2\xi}{\xi+1}\frac{\overline{Q}_{n_0-1}}{P_{n_0}} + \left(1-\gamma_{n_0-1}^{\frac{\xi-1}{2}}\right)\gamma_{n_0-1}\widetilde{B}\right) \\
&+X_6\left(\frac{\eta}{1-\eta}\frac{2\xi}{\xi+1}\frac{\underline{\underline{Q_{n_0+1}}}}{P_{n_0}} + \frac{\xi-1}{\xi+1}\left(1-\gamma_{n_0-1}^{\frac{\xi+1}{2}}\right)\widetilde{b}\right)
\end{aligned} \tag{55}$$

But since $\sum_{k=1}^N \rho^\theta_k = 1$ and $\mu\rho^H_k = \rho^L_k Q_k$, we also know

$$0 = \sum_{k=1}^N \frac{\rho^L_k}{\rho^L_1}(\mu - Q_k) \equiv X_5\left(\mu - \overline{Q}_{n_0-1}\right) + X_6\left(\mu - \underline{Q}_{n_0+1}\right) + (\mu - Q_{n_0}) \tag{56}$$

Letting c and d be the coefficients on X_5 and X_6 (respectively) in (54), and C and D the corresponding coefficients in (55), solve (54) and (55) for X_5 and X_6 in terms of Q_{n_0} as

$$X_5 = \frac{(D-d)\left(\frac{Q_{n_0}}{P_{n_0}}\right)^{\frac{1}{2}(\xi+1)} + \left(D\frac{\xi+1}{\xi-1} - \frac{\xi-1}{\xi+1}\gamma_{n_0-1}d\right)\left(\frac{P_{n_0-1}}{Q_{n_0}}\right)^{\frac{1}{2}(\xi-1)}}{(cD-Cd)} \tag{57}$$

$$X_6 = \frac{(c-C)\left(\frac{Q_{n_0}}{P_{n_0}}\right)^{\frac{1}{2}(\xi+1)} + \left(c\frac{\xi-1}{\xi+1}\gamma_{n_0-1} - \frac{\xi+1}{\xi-1}C\right)\left(\frac{P_{n_0-1}}{Q_{n_0}}\right)^{\frac{1}{2}(\xi-1)}}{(cD-Cd)} \tag{58}$$

Step 2 *$(Q_n)_{n=1}^N$ solves (9) for n_0 iff Q_{n_0} obeys (43) of the Outside-In Recursion.*

We first simplify $\widetilde{B}$ and $\widetilde{b}$ from (53). By (38) and the second expression in (41) at $n = n_0 - 1$,

$$0 = \sum_{k\leq n_0-1}\phi_k f\left(\frac{Q_k}{P_{n_0-1}}\right) = \sum_{k\leq n_0-1}\phi_k\left(\frac{\eta}{1-\eta}\left(\frac{Q_k}{P_n} - \frac{\xi-1}{\xi+1}\right) + \frac{2}{\xi+1}\left(\frac{Q_k}{P_{n_0-1}}\right)^{\frac{\xi+1}{2}}\right)$$

$$\Rightarrow \widetilde{B} \equiv \frac{\sum_{k\leq n_0-1}\phi_k\left(\frac{Q_k}{P_{n_0-1}}\right)^{\frac{1}{2}(\xi+1)}}{\sum_{k\leq n_0-1}\phi_k} = \frac{\eta}{1-\eta}\frac{\xi+1}{2}\left(\frac{\xi-1}{\xi+1} - \frac{\overline{Q}_{n_0-1}}{P_{n_0-1}}\right) \tag{59}$$

And similarly, applying the second equation in (42) to memory state n_0+1 yields

$$\widetilde{b} \equiv \frac{\sum_{k \geq n_0+1} \phi_k \left(\frac{P_{n_0}}{Q_k}\right)^{\frac{1}{2}(\xi-1)}}{\sum_{k \geq i} \phi_k} = \frac{\eta}{1-\eta} \frac{\xi-1}{2} \left(\frac{Q_{n_0+1}}{P_{n_0}} - \frac{\xi+1}{\xi-1}\right) \tag{60}$$

Now, (61) and (62) below give, respectively, the expressions $D-d$ and $D\frac{\xi+1}{\xi-1} - \frac{\xi-1}{\xi+1}\gamma_{n_0-1}d$

$$\frac{\eta}{1-\eta} \frac{2\xi}{\xi+1} \left(\frac{\overline{\underline{Q}}_{n_0+1}}{P_{n_0}} - \frac{\xi+1}{\xi-1}\right) - \left(\frac{\xi+1}{\xi-1} - \beta(\gamma_{n_0-1})\right)\left(1 - \gamma_{n_0-1}^{\frac{\xi-1}{2}}\right)\widetilde{b} \tag{61}$$

$$\frac{\eta}{1-\eta} \frac{2\xi}{\xi-1} \left(\frac{\overline{\underline{Q}}_{n_0+1}}{P_{n_0}} - \frac{\xi-1}{\xi+1}\gamma_{n_0-1}\right) + (1-\gamma_{n_0-1})\,\widetilde{b} \tag{62}$$

And (63) and (64) give, respectively, the expressions $c-C$ and $c\frac{\xi-1}{\xi+1}\gamma_{n_0-1} - \frac{\xi+1}{\xi-1}C$:

$$\frac{\eta}{1-\eta} \frac{2\xi}{\xi+1} \left(\frac{\xi+1}{\xi-1} - \frac{\overline{Q}_{n_0-1}}{P_{n_0-1}}\gamma_{n_0-1}\right) + (1-\gamma_{n_0-1})\,\widetilde{B} \tag{63}$$

$$\frac{\eta}{1-\eta} \frac{2\xi}{\xi-1}\gamma_{n_0-1} \left(\frac{\xi-1}{\xi+1} - \frac{\overline{Q}_{n_0-1}}{P_{n_0-1}}\right) - \left(\frac{\xi+1}{\xi-1} - \beta(\gamma_{n_0-1})\right)\left(1 - \gamma_{n_0-1}^{\frac{\xi-1}{2}}\right)\gamma_{n_0-1}\widetilde{B} \tag{64}$$

First consider (61). By (60) the first term is $\frac{4\xi}{\xi^2-1}\widetilde{b}$, so it simplifies, using (44), as:

$$\frac{D-d}{\frac{\eta}{1-\eta}\frac{\xi+1}{2}\left(1 - \left(\frac{\xi-1}{\xi+1}\right)^2 \gamma_{n_0-1}\right)} = \gamma_{n_0-1}^{\frac{1}{2}(\xi-1)} \left(\frac{\underline{Q}_{n_0+1}}{P_{n_0}} - \frac{\xi+1}{\xi-1}\right) \equiv X_1 \tag{65}$$

And plug $\widetilde{B} = \frac{\eta}{1-\eta}\frac{\xi+1}{2}\left(\frac{-4\xi}{\xi^2-1} + \frac{\xi+1}{\xi-1} - \frac{\overline{Q}_{n_0-1}}{P_{n_0-1}}\right)$ from (59) into (63) and simplify to get:

$$\frac{c-C}{\frac{\eta}{1-\eta}\frac{\xi+1}{2}\left(1 - \left(\frac{\xi-1}{\xi+1}\right)^2 \gamma_{n_0-1}\right)} = \left(\frac{\xi+1}{\xi-1} - \frac{\overline{Q}_{n_0-1}}{P_{n_0-1}}\right) \equiv X_3 \tag{66}$$

By similar computations, plugging (59) into (64) and (60) into (62) and collecting terms,

$$\frac{D\frac{\xi+1}{\xi-1} - \frac{\xi-1}{\xi+1}\gamma_{n_0-1}d}{\frac{\eta}{1-\eta}\frac{\xi+1}{2}\left(1 - \left(\frac{\xi-1}{\xi+1}\right)^2 \gamma_{n_0-1}\right)} \equiv X_2 \text{ and } \frac{c\frac{\xi-1}{\xi+1}\gamma_{n_0-1} - \frac{\xi+1}{\xi-1}C}{\frac{\eta}{1-\eta}\frac{\xi+1}{2}\left(1 - \left(\frac{\xi-1}{\xi+1}\right)^2 \gamma_{n_0-1}\right)} \equiv X_4 \tag{67}$$

$$\text{while } \frac{cD - Cd}{\frac{\eta}{1-\eta}\frac{\xi+1}{2}\left(1 - \left(\frac{\xi-1}{\xi+1}\right)^2 \gamma_{n_0-1}\right)} = \frac{\eta}{1-\eta}\frac{\xi-1}{2}\left(X_2X_3 - X_1X_4\right) \tag{68}$$

With these, expressions (57) and (58) directly simplify to those in (45), while plugging (45) into (56) and multiplying by $\frac{\eta}{1-\eta}\frac{\xi-1}{2}\left(X_2X_3 - X_1X_4\right)/\left(\mu - \overline{Q}_{n_0-1}\right)$ yields (43) for Q_{n_0}. $\square$

C.4 Existence, Uniqueness, and Bounds for $n = n_0$

For uniqueness and to prove that $Q_{n_0} \in [P_{n_0-1}, P_{n_0}]$, we prove that expression (43) for Q_{n_0} is (i) decreasing in Q_{n_0}, (ii) positive at $Q_{n_0} = P_{n_0-1}$, and (iii) negative at $Q_{n_0} = P_{n_0}$.

Step 1: Proof of (i). Consider the coefficient in (43) on $\mu - Q_{n_0}$. This is positive, since Bayes consistency implies $\overline{Q}_{n_0-1} \le \mu$, while by $\gamma_{n_0-1} \le 1 < \xi$ the X_i's in (44) obey $X_1 < \frac{Q_{n_0+1}}{P_{n_0}} - 1 < X_2$ and $X_4 < 1 - \frac{\overline{Q}_{n_0-1}}{P_{n_0-1}} < X_3$, thus $X_2X_3 - X_1X_4 > 0$. Next take the coefficient on $\left(\frac{Q_{n_0}}{P_{n_0}}\right)^{\frac{\xi+1}{2}}$. We claim that it has the following upper bound, which is negative by $P_{n_0} \ge \mu$:

$$X_1 - \frac{Q_{n_0+1} - \mu}{\mu - \overline{Q}_{n_0-1}} X_3 < \frac{\left(\frac{Q_{n_0+1}}{P_{n_0}} - \frac{\xi+1}{\xi-1}\right) - \frac{Q_{n_0+1}-\mu}{\mu-\overline{Q}_{n_0-1}}\left(\frac{\xi+1}{\xi-1} - \frac{\overline{Q}_{n_0-1}}{P_{n_0}}\right)}{\frac{(\xi+1)^2}{4\xi}\left(1 - \left(\frac{\xi-1}{\xi+1}\right)^2 \gamma_{n_0-1}\right)} = \frac{\left(\frac{\mu}{P_{n_0}} - \frac{\xi+1}{\xi-1}\right)\left(\frac{Q_{n_0+1}-\overline{Q}_{n_0-1}}{\mu-\overline{Q}_{n_0-1}}\right)}{\frac{(\xi+1)^2}{4\xi}\left(1 - \left(\frac{\xi-1}{\xi+1}\right)^2 \gamma_{n_0-1}\right)}$$

For the middle bound: By (53) we know $\widetilde{b} > 0$ and $\widetilde{B} > 0$, so (recalling $\beta < 1$) setting $\widetilde{b} = 0$ gives an upper bound on $D - d$ from (61); plug this into (65) for an upper bound on X_1 (this is the first ratio in our bound above). Similarly, set $\widetilde{B} = 0$ for a lower bound on $c - C$ in (63); plug this into (66) for a lower bound on X_3 (the final ratio in our bound). Finally, the coefficient in (43) on $\left(\frac{P_{n_0-1}}{Q_{n_0}}\right)^{\frac{\xi-1}{2}}$ has the following lower bound, positive by $P_{n_0-1} \le \mu$:

$$X_2 - \frac{Q_{n_0+1} - \mu}{\mu - \overline{Q}_{n_0-1}} X_4 > \frac{\left(\frac{Q_{n_0+1}}{P_{n_0-1}} - \frac{\xi-1}{\xi+1}\right) - \frac{Q_{n_0+1}-\mu}{\mu-\overline{Q}_{n_0-1}}\left(\frac{\xi-1}{\xi+1} - \frac{\overline{Q}_{n_0-1}}{P_{n_0-1}}\right)}{\frac{1}{\gamma_{n_0-1}}\frac{\xi^2-1}{4\xi}\left(1 - \left(\frac{\xi-1}{\xi+1}\right)^2 \gamma_{n_0-1}\right)} = \frac{\left(\frac{\mu}{P_{n_0}} - \frac{\xi-1}{\xi+1}\gamma_{n_0-1}\right)\frac{Q_{n_0+1}-\overline{Q}_{n_0-1}}{\mu-\overline{Q}_{n_0-1}}}{\frac{\xi^2-1}{4\xi}\left(1 - \left(\frac{\xi-1}{\xi+1}\right)^2 \gamma_{n_0-1}\right)}$$

For this, obtain a lower bound on X_2 by setting $\widetilde{b} = 0$ in (62) and plugging into (67); and an upper bound on X_4 by setting $\widetilde{B} = 0$ for an upper bound on (64), then plug into (67). $\square$.

Step 2: Proof of (ii). By Step 1, the final term in (43) is a positive coefficient multiplied by $\mu - Q_{n_0}$, positive here ($Q_{n_0} = P_{n_0-1}$) by initial state restriction $P_{n_0-1} \le \mu$. So it suffices to prove that the sum of the first two terms is positive at $Q_{n_0} = P_{n_0-1}$, i.e. that

$$\left(X_1 - \frac{(\overline{Q}_{n_0+1} - \mu)}{\mu - \overline{Q_{n_0-1}}} X_3\right)\gamma_{n_0-1}^{\frac{1}{2}(\xi+1)} + X_2 - \frac{(\overline{Q}_{n_0+1} - \mu)}{\mu - \overline{Q}_{n_0-1}} X_4 > 0 \tag{69}$$

Using (44), the LHS of this expression is

$$\left(\frac{\underline{Q}_{n_0+1}}{P_{n_0}} - \frac{\xi+1}{\xi-1}\right)\left(\gamma_{n_0-1}^{\xi} + \frac{\xi+1}{\xi-1}\right) + \frac{4\xi}{(\xi-1)^2} - \frac{\underline{Q}_{n_0+1}-\mu}{\mu-\overline{Q}_{n_0-1}}\frac{2\xi}{\xi-1}\gamma_{n_0-1}^{\frac{\xi+1}{2}}\left(1-\frac{\mu}{P_{n_0-1}}+\frac{\mu-\overline{Q}_{n_0-1}}{P_{n_0-1}}\right)$$

Since $P_{n_0-1} \leq \mu$, replacing $1-\frac{\mu}{P_{n_0-1}}$ in the final term by 0 gives lower bound:

$$\begin{aligned} & \left(\frac{\underline{Q}_{n_0+1}}{P_{n_0}} - \frac{\xi+1}{\xi-1}\right)\left(\gamma_{n_0-1}^{\xi} + \frac{\xi+1}{\xi-1}\right) + \frac{4\xi}{(\xi-1)^2} - \frac{2\xi}{\xi-1}\gamma_{n_0-1}^{\frac{\xi-1}{2}}\left(\frac{\underline{Q}_{n_0+1}-\mu}{P_{n_0}}\right) \\ = \; & \left(\frac{\underline{Q}_{n_0+1}}{P_{n_0}} - \frac{\xi+1}{\xi-1}\right)\left(\gamma_{n_0-1}^{\xi} + \frac{\xi+1}{\xi-1} - \frac{2\xi}{\xi-1}\gamma_{n_0-1}^{\frac{\xi-1}{2}}\right) + \frac{2\xi}{\xi-1}\left[\frac{2}{\xi-1} - \gamma_{n_0-1}^{\frac{\xi-1}{2}}\left(\frac{\xi+1}{\xi-1} - \frac{\mu\gamma_{n_0-1}}{P_{n_0-1}}\right)\right] \end{aligned}$$

The first bracketed term is proportional to $\widetilde{b} > 0$ by (60), (53). The second bracketed term is also positive: It decreases in γ_{n_0-1}, with derivative $\left(\gamma_{n_0-1}^{\frac{\xi+1}{2}} - 1\right)\xi\gamma_{n_0-1}^{\frac{\xi-3}{2}}$ thus is at least its value (0) at $\gamma_{n_0-1} = 1$. And the final term is positive: it's at least its value if we replace P_{n_0-1} by upper bound μ, which simplifies to $\left(\frac{1}{\beta}-1\right)\left(1-\gamma_{n_0-1}^{\frac{\xi+1}{2}}\right) > 0$ by $\beta < 1$. □.

Step 3: Proof of (iii). By Step 1, the final term in (43) is a positive coefficient multiplied by $\mu - Q_{n_0}$, nonpositive here ($Q_{n_0} = P_{n_0}$) by initial state restriction $P_{n_0} \geq \mu$. So it suffices to prove that the sum of the first two terms is negative at $Q_{n_0} = P_{n_0}$, i.e. that

$$X_1 - \frac{(\overline{Q}_{n_0+1}-\mu)}{\mu-\overline{Q}_{n_0-1}}X_3 + \left(X_2 - \frac{(\overline{Q}_{n_0+1}-\mu)}{\mu-\overline{Q}_{n_0-1}}X_4\right)\gamma_{n_0-1}^{\frac{1}{2}(\xi-1)} < 0$$

The LHS, using (44) for the first line, then $\gamma_{n_0-1} \equiv \frac{P_{n_0-1}}{P_{n_0}}$ and $P_{n_0} \geq \mu$ for the inequality, is:

$$\begin{aligned} & \left(\frac{\underline{Q}_{n_0+1}}{P_{n_0}} - 1\right)\frac{2\xi}{\xi-1}\gamma_{n_0-1}^{\frac{\xi-1}{2}} - \left(\frac{\underline{Q}_{n_0+1}-\mu}{\mu-\overline{Q}_{n_0-1}}\right)\left(\frac{\xi+1}{\xi-1} - \frac{\overline{Q}_{n_0-1}}{P_{n_0-1}} + \gamma_{n_0-1}^{\xi}\left(1-\frac{\xi+1}{\xi-1}\frac{\overline{Q}_{n_0-1}}{P_{n_0-1}}\right)\right) \\ < \; & \left(\frac{\underline{Q}_{n_0+1}-\mu}{P_{n_0-1}}\right)\frac{2\xi}{\xi-1}\gamma_{n_0-1}^{\frac{\xi+1}{2}} - \left(\frac{\underline{Q}_{n_0+1}-\mu}{\mu-\overline{Q}_{n_0-1}}\right)\left(\frac{\xi+1}{\xi-1} - \frac{\overline{Q}_{n_0-1}}{P_{n_0-1}} + \gamma_{n_0-1}^{\xi}\left(1-\frac{\xi+1}{\xi-1}\frac{\overline{Q}_{n_0-1}}{P_{n_0-1}}\right)\right) \end{aligned}$$

Multiplying by $(\mu - \overline{Q}_{n_0-1})/\left(\underline{Q}_{n_0+1} - \mu\right)$, it suffices to show that the following is negative:

$$\begin{aligned} & \left(\frac{\mu-\overline{Q}_{n_0-1}}{P_{n_0-1}}\right)\frac{2\xi}{\xi-1}\gamma_{n_0-1}^{\frac{\xi+1}{2}} - \left(\frac{\xi+1}{\xi-1} - \frac{\overline{Q}_{n_0-1}}{P_{n_0-1}} + \gamma_{n_0-1}^{\xi}\left(1-\frac{\xi+1}{\xi-1}\frac{\overline{Q}_{n_0-1}}{P_{n_0-1}}\right)\right) \\ = \; & \left(\frac{\mu\gamma_{n_0-1}^{\frac{\xi-1}{2}}}{P_{n_0}} - \frac{\xi-1}{\xi+1}\gamma_{n_0-1}^{\frac{\xi+1}{2}} - \frac{2}{\xi+1}\right)\frac{2\xi}{\xi-1} + \left(\frac{\xi-1}{\xi+1} - \frac{\overline{Q_{n_0-1}}}{P_{n_0-1}}\right)\left(\frac{2\xi}{\xi-1}\gamma_{n_0-1}^{\frac{\xi+1}{2}} - 1 - \frac{\xi+1}{\xi-1}\gamma_{n_0-1}^{\xi}\right) \end{aligned}$$

The final bracketed expression is increasing in γ_{n_0-1}, with derivative $\xi\frac{\xi-1}{\xi+1}\gamma^{\frac{\xi-1}{2}}\left(1-\gamma^{\frac{\xi-1}{2}}\right)$, and thus is at most its value at upper bound (0) $\gamma_{n_0-1}=1$. The second last bracketed term is proportional to $\widetilde{B}>0$ from (59). And the first bracketed term is decreasing in $P_{n_0}\geq\mu$, thus at most its value at $P_{n_0}=\mu$, which simplifies to $(1-\gamma^{\frac{1}{2}(\xi-1)})(\beta-1)<0$ (by $\beta<1$). □

C.5 Computing Vectors ρ^H and ρ^L

To derive equation (45): The final paragraph of Appendix C.3 derived the $X_5=(\sum_{k\leq n_0-1}\rho^L_k)/\rho^L_{n_0}$ and $X_6=(\sum_{k\geq n_0+1}\rho^L_k)/\rho^L_{n_0}$ expressions. For $\rho^L_{n_0}$ use $1=\sum\rho^L_k\equiv\rho^L_{n_0}(1+X_5+X_6)$. For ρ^L_1 and ρ^L_n, use the definitions $\phi_j=\rho^L_j/\rho^L_1$ and $\Phi_j\ \rho^H_j/\rho^H_N$ along with Bayes rationality to rewrite X_5 and X_6 as $X_5\equiv\rho^L_1\left(\sum_{j\leq n_0-1}\phi_j\right)/\rho^L_{n_0}$ and $X_6\equiv\rho^L_N\left(\sum_{j\geq n_0+1}\phi_j\frac{Q_N}{Q_j}\right)/\rho^L_{n_0}$. □

D Optimal Memories When $N=2$

D.1 Setting Up the Optimality Conditions

To derive (16): By Part 1 of Definition 1, beliefs in memory state 1 are simply $Q_1=P_1\underline{x}$. In $n_0=2$, we simplify (43) as follows, noting that the highest cutoff (here P_2) is ∞, thus the first of its three expressions vanishes as do X_1 and X_4 from (44) using $\gamma_{n_0-1}\equiv P_1/P_2=0$:

$$\left(\frac{P_1}{Q_2}\right)^{\frac{\xi-1}{2}}\left(\frac{\mu-P_1\underline{x}}{Q_2-\mu}\right)=\frac{\eta}{1-\eta}\frac{\xi-1}{2}X_3,\ \text{where } X_3=\frac{\xi+1}{\xi-1}-\underline{x}\ \text{by (44)} \tag{70}$$

To derive (17), we now compute the post-amnesia payoffs. By (12) and (10), these solve

$$\begin{pmatrix}\frac{\eta}{1-\eta}+\lambda^\theta_{1,2} & -\lambda^\theta_{1,2}\\ -\lambda^\theta_{2,1} & \frac{\eta}{1-\eta}+\lambda^\theta_{2,1}\end{pmatrix}\begin{pmatrix}\frac{1-\eta}{\eta}\nu^\theta_1\\ \frac{1-\eta}{\eta}\nu^\theta_2\end{pmatrix}=\begin{pmatrix}w^\theta_1\\ w^\theta_2\end{pmatrix} \tag{71}$$

$$\text{where } w^H_i\equiv\Pi\left(1-F^H\left(\log\frac{\pi}{\Pi Q_i}\right)\right)\ \text{and}\ w^L_i=\pi F^L\left(\log\frac{\pi}{\Pi Q_i}\right) \tag{72}$$

Using Cramer's rule to solve (71) for $\nu^\theta_2-\nu^\theta_1$, letting $\Gamma^\theta\equiv\frac{\eta}{1-\eta}+\lambda^\theta_{1,2}+\lambda^\theta_{2,1}$, we obtain:

$$\frac{1-\eta}{\eta}(\nu^\theta_2-\nu^\theta_1)=\frac{\frac{\eta}{1-\eta}(w^\theta_2-w^\theta_1)}{\Gamma^\theta}$$

So the indifference FOC, namely $1=P_1(\nu^H_2-\nu^H_1)/(\nu^L_1-\nu^L_2)$, becomes:

$$1=P_1\frac{\Gamma^L}{\Gamma^H}\frac{w^H_2-w^H_1}{w^L_1-w^L_2} \tag{73}$$

To simplify, plug (15) and (14) into the Bayes consistency condition for memory state 1:

$$Q_1 = \mu \frac{\rho_1^H}{\rho_1^L} = \mu \frac{\lambda_{2,1}^H}{\lambda_{2,1}^L} \frac{\Gamma^L}{\Gamma^H} = \mu \frac{\xi - 1}{\xi + 1} \frac{P_1}{Q_2} \frac{\Gamma^L}{\Gamma^H} \Rightarrow P_1 \frac{\Gamma^L}{\Gamma^H} = \frac{\xi + 1}{\xi - 1} \frac{Q_1 Q_2}{\mu} \tag{74}$$

Plugging (74) and (72) into (73), we obtain the following FOC:

$$1 = \frac{1}{\mu} \frac{\xi + 1}{\xi - 1} \frac{\Pi}{\pi} Q_1 Q_2 \frac{F^H\left(\log \frac{\pi}{\Pi Q_1}\right) - F^H\left(\log \frac{\pi}{\Pi Q_2}\right)}{F^L\left(\log \frac{\pi}{\Pi Q_1}\right) - F^L\left(\log \frac{\pi}{\Pi Q_2}\right)} \tag{75}$$

Evaluating this using F^θ from (5) yields text FOC (17), noting that optimally, $Q_1 < \pi/\Pi < Q_2$: This is immediate for Q_2 by $\pi < \Pi\mu$ and Bayes consistency ($Q_1 < \mu < Q_2$), and if we also had $Q_1 > \pi/\Pi$, then by (5) $F^H(\log \frac{\pi}{\Pi Q_i}) = \frac{\xi-1}{2\xi}(\frac{\pi}{\Pi Q_i})^{\frac{\xi+1}{2}}$ for $i = 1, 2$, swapping $\xi+1$ with $\xi-1$ for F^L. Then (75) fails; the final ratio exceeds $\frac{\xi-1}{\xi+1}\frac{\pi}{\Pi Q_1}$ thus the RHS exceeds $Q_2/\mu > 1$.

D.2 Preliminary Comparative Statics

Rewrite (17) as follows, with $x \equiv \pi/\Pi Q_1$ and $y \equiv \pi/\Pi Q_2$, where we just established $x > 1 >$y:

$$1 = \frac{\xi + 1}{\xi - 1} \frac{\pi}{\mu \Pi} \frac{I}{xy}, \text{ where } I = \frac{1 - \frac{\xi+1}{2\xi} x^{-\frac{\xi-1}{2}} - \frac{\xi-1}{2\xi} y^{\frac{\xi+1}{2}}}{1 - \frac{\xi-1}{2\xi} x^{-\frac{\xi+1}{2}} - \frac{\xi+1}{2\xi} y^{\frac{\xi-1}{2}}} \tag{76}$$

Lemma D.1 *$I_x \geq 0$, $I_y \geq 0$, and $y \leq I \leq x$.*

PROOF: Directly differentiating expression I in (76) yields

$$x \frac{I_x}{I} = \frac{x^{-\frac{1}{2}(\xi+1)}(x - I)}{\frac{2}{\xi-1}\left(1 - x^{-\frac{1}{2}(\xi-1)}\right) + \frac{2}{\xi+1}\left(1 - y^{\frac{1}{2}(\xi+1)}\right)} \tag{77}$$

$$y \frac{I_y}{I} = \frac{y^{\frac{1}{2}(\xi+1)}\left(\frac{I}{y} - 1\right)}{\frac{2}{\xi-1}\left(1 - x^{-\frac{1}{2}(\xi-1)}\right) + \frac{2}{\xi+1}\left(1 - y^{\frac{1}{2}(\xi+1)}\right)} \tag{78}$$

It is immediate from (77) and (78) the third Lemma D.1 claim implies the first two, so it suffices to show $y \leq I \leq x$. Suppose (toward a contradiction) $I < y < x$. Then $I_y < 0 < I_x$ (by (77) and (78)), so since $x > 1 > y$, setting $x = y = 1$ yields a lower bound on I:

$$I \geq \frac{1 - \frac{\xi+1}{2\xi} - \frac{\xi-1}{2\xi} y^{\frac{1}{2}(\xi+1)}}{1 - \frac{\xi-1}{2\xi} - \frac{\xi+1}{2\xi} y^{\frac{1}{2}(\xi-1)}} \geq \lim_{y \uparrow 1} \frac{-\frac{\xi-1}{2\xi}\frac{\xi+1}{2y} y^{\frac{1}{2}(\xi+1)}}{-\frac{\xi+1}{2\xi}\frac{\xi-1}{2y} y^{\frac{1}{2}(\xi-1)}} = y$$

Contradicting $I < y$. Similarly if $I > x > y$, then the above expressions yield $I_x < 0 < I_y$, so $x > 1 > y$ implies that I is at most its value at $x = y = 1$. First evaluating I at $x = 1$ and then taking limits as $y \to 1$ yields upper bound x, contradicting $I > x$. $\square$

Lemma D.2 $xI_x + yI_y < I$

PROOF: We wish to prove that the sum of the expressions in (77) and (78) is smaller than 1, i.e. (multiplying through by their common denominator) that

$$x^{-\frac{1}{2}(\xi+1)}(x - I) + y^{\frac{1}{2}(\xi+1)}\left(\frac{I}{y} - 1\right) < \frac{2}{\xi - 1}\left(1 - x^{-\frac{1}{2}(\xi-1)}\right) + \frac{2}{\xi + 1}\left(1 - y^{\frac{1}{2}(\xi+1)}\right)$$

Adding $(1 - x^{-\frac{1}{2}(\xi-1)}) - \left(1 - y^{\frac{1}{2}(\xi+1)}\right)$ to both sides, this becomes

$$\left[\left(1 - x^{-\frac{1}{2}(\xi+1)}\right) - \left(1 - y^{\frac{1}{2}(\xi-1)}\right)\right] I < \frac{\xi + 1}{\xi - 1}\left(1 - x^{-\frac{1}{2}(\xi-1)}\right) - \frac{\xi - 1}{\xi + 1}\left(1 - y^{\frac{1}{2}(\xi+1)}\right) \tag{79}$$

If both the LHS and RHS of (79) are negative, it rearranges to the following lower bound on I (where the numerator and denominator are positive in this case):

$$I \geq \frac{\frac{\xi-1}{\xi+1}(1 - y^{\frac{\xi+1}{2}}) - \frac{\xi+1}{\xi-1}(1 - x^{-\frac{\xi-1}{2}})}{(1 - y^{\frac{\xi-1}{2}}) - (1 - x^{-\frac{\xi+1}{2}})} \tag{80}$$

If both the LHS and RHS of (79) are positive, then it instead rearranges to the following upper bound on I, again arranged so numerator and denominator are both positive:

$$I \leq \frac{\frac{\xi+1}{\xi-1}(1 - x^{-\frac{\xi-1}{2}}) - \frac{\xi-1}{\xi+1}(1 - y^{\frac{\xi+1}{2}})}{(1 - x^{-\frac{\xi+1}{2}}) - (1 - y^{\frac{\xi-1}{2}})} \tag{81}$$

We also must rule out the possibility that only the RHS of (79) is negative, while it holds trivially if only the LHS is negative. Lastly, from slightly rearranging (76), we actually have

$$I = \frac{\frac{\xi-1}{\xi+1}\left(1 - y^{\frac{1}{2}(\xi+1)}\right) + \left(1 - x^{-\frac{1}{2}(\xi-1)}\right)}{\left(1 - y^{\frac{1}{2}(\xi-1)}\right) + \frac{\xi-1}{\xi+1}\left(1 - x^{-\frac{1}{2}(\xi+1)}\right)} \tag{82}$$

Comparing these expressions, it suffices to prove the following inequality:

$$\frac{\frac{\xi-1}{\xi+1}\left(1 - y^{\frac{1}{2}(\xi+1)}\right)}{\left(1 - y^{\frac{1}{2}(\xi-1)}\right)} < \frac{\xi + 1}{\xi - 1}\frac{\left(1 - x^{-\frac{1}{2}(\xi-1)}\right)}{\left(1 - x^{-\frac{1}{2}(\xi+1)}\right)} \tag{83}$$

For then the bound in (80) is below the smaller LHS expression in (83), the bound in (81) exceeds the larger RHS expression in (83), and I from (82) lies between the two bounds in (83). And (83) also ensures that if the RHS of (79) is negative, then so is the LHS.

Finally we prove that (83) indeed holds, namely that its LHS is at most 1 while its RHS is at least 1. Recall that $x \geq 1 \geq y$. First differentiate to obtain that the LHS of (83) is increasing in y whenever the following (derivative in y) is positive:

$$-\frac{\xi+1}{2y}y^{\frac{1}{2}(\xi+1)} + \frac{\xi-1}{2y}y^{\frac{1}{2}(\xi-1)}\frac{\left(1-y^{\frac{1}{2}(\xi+1)}\right)}{\left(1-y^{\frac{1}{2}(\xi-1)}\right)} > 0 \Leftrightarrow \frac{\xi-1}{\xi+1}\frac{\left(1-y^{\frac{1}{2}(\xi+1)}\right)}{\left(1-y^{\frac{1}{2}(\xi-1)}\right)} > y$$

I.e. whenever it exceeds y. So either the LHS is below $y \leq 1$, or it's increasing in y and thus at most its value as $y \uparrow 1$, which by l'hopital's rule is 1. And by a symmetric argument, the RHS of (83) is increasing in x whenever it is smaller than x. So either it exceeds $x \geq 1$, or it's increasing in x and thus at least its limit as $x \downarrow 1$, which by l'hopital's rule is 1. □

Lemma D.3 *The RHS of* (76) *is decreasing in* π/Π *and in* μ*.*

Proof: The μ part is obvious. For π/Π, recalling that $x = \pi/\Pi Q_1$ and $y = \pi/\Pi Q_2$ depend on it, the derivative of the RHS of (76) in π/Π is

$$\begin{aligned} & \frac{I}{xy} + \frac{\pi}{\Pi}\left[\frac{\partial}{\partial x}\left(\frac{I}{xy}\right)\cdot\frac{dx}{d(\pi/\Pi)} + \frac{\partial}{\partial y}\left(\frac{I}{xy}\right)\cdot\frac{dy}{d(\pi/\Pi)}\right] \\ = \ & \frac{I}{xy} + \frac{\pi}{\Pi}\left[\frac{I_x - \frac{I}{x}}{xy}\frac{1}{Q_1} + \frac{I_y - \frac{I}{xy}x}{xy}\frac{1}{Q_2}\right] \\ = \ & \frac{I}{xy} + \left[\frac{xI_x - I}{xy} + \frac{yI_y - I}{xy}\right], \text{using } \frac{1}{Q_1} \equiv \frac{\Pi x}{\pi} \text{and } \frac{1}{Q_2} \equiv \frac{\Pi y}{\pi} \end{aligned}$$

This is negative iff $xI_x + yI_y - I < 0$, which we proved in Lemma D.2. □

Lemma D.4 *The LHS of* (70) *falls in* Q_2 *and rises in* P_1*;* $\underline{x}$ *and the RHS of* (70) *rise in* η*.*

Proof: The LHS claim regarding Q_2 follows by immediate inspection, since belief consistency demands $Q_2 > \mu$. For P_1, differentiate to find that the LHS of (70) increases in P_1 iff $\frac{\mu}{P_1} > \frac{\xi+1}{\xi-1}\underline{x}$; this holds since by (38) the root $\underline{x}$ of f is smaller than $\frac{\xi-1}{\xi+1}$, while optimality of $n_0 = 2$ implies $P_1 < \mu$. For $\underline{x}$, rewrite the equation $f(\underline{x}) = 0$ as follows:

$$\frac{\eta}{1-\eta}\underline{x}^{-\frac{1}{2}(\xi+1)}\left(\frac{\xi-1}{2} - \frac{\xi+1}{2}\underline{x}\right) = 1 \tag{84}$$

Since the LHS clearly rises with η and falls with $\underline{x}$ (recalling $\underline{x} < \frac{\xi-1}{\xi+1}$), it can only remain constant at 1 if $\underline{x}$ rises with η. And finally, using using (84) to replace $\eta/(1-\eta)$, the RHS of (70) is proportional to $\underline{x}^{\frac{1}{2}(\xi+1)} \frac{\xi+1-(\xi-1)\underline{x}}{\xi-1-(\xi+1)\underline{x}}$; this rises in $\underline{x}$ which we just showed rises with η.

D.3 Completing the Proof of Proposition 3

Step 1 *The RHS of the FOC* (17) *increases in both* Q_1 *and* Q_2*.*

Recalling $x \equiv \pi/\Pi Q_1$ and $y \equiv \pi/\Pi Q_2$, it suffices to show that the RHS of equivalent FOC (76) falls in x and y, i.e. that I/xy falls in x and y. This follows from combining the Lemma D.1 result that xI_x/I and yI_y/I are both positive, with the Lemma D.2 result that they sum to less than 1. Thus $xI_x/I < 1$, so I/x falls in x, and similarly I/y falls in y.

Step 2 *If* η *rises, the Bayesian constraint* (70) *requires that either* P_1 *rise or* Q_2 *fall (or both). Also, fixing* η *and* ξ*, an increase in* P_1/μ *leads to an increase in both* Q_1/μ *and* Q_2/μ*.*

For the first assertion, we know from Lemma D.4 that an increase in η leads to an increase in the RHS of (70), and also an increase in $\underline{x}$ which (by immediate inspection, recalling $Q_1 < P_1 < Q_2$) lowers the LHS of (70). So it must be accompanied by another change that increases the LHS of (70). By Lemma D.4, this is to either raise P_1 or lower Q_2. For the second assertion, Lemma D.4 directly shows that with no change in μ, Q_2 and P_1 must move the same direction to hold the LHS of (70) constant, and Q_1 does too since $Q_1 = P_1\underline{x}$. If μ also changes, simply divide numerator and denominator in the LHS of (70) by μ, and apply identical logic to P_1/μ, Q_1/μ, and Q_2/μ.

Step 3 (Existence and Uniqueness) *There is a unique solution to* (76) *and* (70)

For existence, observe that for any P_1, there is a unique Q_2 satisfying belief constraint (70) by the Intermediate Value Theorem (IVT), since the LHS is decreasing (and continuous) in Q_2 by Lemma D.4, and tends to ∞ as $Q_2 \to 1$, and 0 as $Q_2 \to \infty$. So, for any Q_1, let $Q_2^*(Q_1)$ be the Q_2 that solves (70). Next, notice that for *any* Q_2, the RHS of (76) tends to zero as $x \to \infty$ i.e. $Q_1 = 0$, while we established at the end of Section D.1 that it is too large at $x = 1 \Leftrightarrow Q_1 = \pi/\Pi$. So again by the IVT, a solution Q_1 to (76) (with $Q_2 = Q_2^*(Q_1)$) exists. For uniqueness, consider a solution (Q_1, Q_2) to (76) and (70). If we increase $Q_1 = P_1\underline{x}$, Q_2 must also increase by Step 2. But then (76) fails: both changes increase its RHS by Step 1.

Step 4 (Comparative Statics in π/Π and μ) *The optimal* P_1 *rises with* π/Π *and* μ*.*

By Lemma D.3, the RHS of (76) falls in π/Π and in μ, so must be accompanied by an offsetting increase. By Step 1 of this proof, this requires increasing Q_1 and/or Q_2, but by Step 2, Q_1/μ, Q_2/μ, and P_1/μ all move in the same direction. Thus P_1,Q_1, and Q_2 all rise.

Step 5 (Comparative Static in η) *As η rises, Q_1 rises and Q_2 falls.*

Since increasing η has no direct effect on FOC (76), Q_1 and Q_2 must move in opposite directions. This cannot happen if Q_2 rises; for then by Step 2, P_1 also rises, as does $\underline{x}$ by the second assertion in Lemma D.4, thus so does $Q_1 = P_1\underline{x}$. □.

D.4 Payoffs

The value in (11) is π times

$$\frac{\mu\Pi}{\pi}\left[\rho_1^H\frac{w_1^H}{\Pi}+\rho_2^H\frac{w_2^H}{\Pi}\right]+\left[\rho_1^L\frac{w_1^L}{\pi}+\rho_2^L\frac{w_2^L}{\pi}\right] \tag{85}$$

Since probabilities sum to 1 and $Q_i = \mu\rho_i^H/\rho_i^L$, $\mu(\rho_1^L+\rho_2^L) = \mu = \rho_1^L Q_1 + \rho_2^L Q_2$. Solving,

$$\rho_1^H = \frac{Q_1\left(Q_2-\mu\right)}{\mu(Q_2-Q_1)} \text{ and } \rho_2^H = \frac{Q_2(\mu-Q_1)}{\mu(Q_2-Q_1)}, \rho_1^L = \mu\frac{\rho_1^H}{Q_1}, \rho_2^L = \mu\frac{\rho_2^H}{Q_2}$$

Plugging these into (85), along with (72) and (5) for w_i^θ, recalling $Q_1 < \frac{\pi}{\Pi} < Q_2$, it becomes

$$\frac{Q_2-\mu}{Q_2-Q_1}\left(\frac{1}{\xi}\left(\frac{\Pi Q_1}{\pi}\right)^{\frac{1}{2}(\xi+1)}+1\right)+\frac{\mu-Q_1}{Q_2-Q_1}\left(\frac{\Pi Q_2}{\pi}+\frac{1}{\xi}\left(\frac{\pi}{\Pi Q_2}\right)^{\frac{1}{2}(\xi-1)}\right) \tag{86}$$

E Omitted Value Function Proofs for Section 6

A. Proof of Lemma 3: Strict Convexity.

Step 1 *Dug Benefits from Delay if the Next Shock is a Decision Shock:*

$$p\Pi(1-F^H(\hat{\ell}-\ell(p))+(1-p)\pi F^L(\hat{\ell}-\ell(p))-u(p)>0 \tag{87}$$

If $p \geq \hat{p}$ (symmetric steps apply if $p \leq \hat{(}p)$), (87) reduces using $u(p) = p\Pi$ and $P \equiv \frac{p}{1-p}$ as:

$$-p\Pi\left(1-F^H\log\left(\frac{\pi}{\Pi P}\right)\right)+(1-p)\pi\left(1-F^L\log\left(\frac{\pi}{\Pi P}\right)\right)>0 \Leftrightarrow \frac{P\Pi}{\pi}\left(\frac{F^H\log\left(\frac{\pi}{\Pi P}\right)}{F^L\log\left(\frac{\pi}{\Pi P}\right)}\right)<1$$

This holds by (5): for $P \geq \frac{\pi}{\Pi}$, the final bracketed ratio is $\frac{\xi-1}{\xi+1}\frac{\pi}{\Pi P}$. □.

Step 2 *Dug Prefers Amnesia Shocks be Delayed:* $\sum_n \lambda_n^\theta(p)\nu_n^\theta \geq \nu_k^\theta \quad$ *for* $p \in [p_{k-1}, p_k)$

By (5), F^H increases in α, while ν_n^H increases in n by Lemma 2. Thus, fixing $\mathbf{p}$:

$$\sum_n \frac{\partial \lambda_n^H(p)}{\partial \alpha}\nu_n^H = \sum_n \frac{\partial F^H(\ell(p_n) - \ell(p))}{\partial \alpha}(\nu_n^H - \nu_{n+1}^H) < 0$$

And if p is in memory state k, then $\lim_{\alpha\to\infty} \lambda_k^H(p) = 1 \Rightarrow \sum_n \left(\lim_{\alpha\to\infty} \lambda_n^H(p)\right)\nu_n^H = \nu_k^H$. Altogether, for fixed $\mathbf{p}$, the sum $\sum_n \left(\lim_{\alpha\to\infty} \lambda_n^H(p)\right)\nu_n^H$ converges to ν_k^H from above. Similar steps apply in state $\theta = L$. $\square$.

Step 3 V_n *is strictly convex for each* n.

Subtracting (22) from a $(1-p, p)$ weighted average of (18) and (19) we discover:

$$\begin{aligned} 0 &= \eta\left(p\Pi(1 - F^H(\hat{\ell} - \ell(p)) + (1-p)\pi F^L(\hat{\ell} - \ell(p)) - u(p)\right) \\ &\quad +(1-\eta)\left(\sum_n (p\lambda_n^H(p)\nu_n^H + (1-p)\lambda_n^L(p)\nu_n^L) - \nu_n(p)\right) - \frac{2p^2(1-p)^2}{(\alpha+\delta)\sigma^2}V_n''(p) \end{aligned}$$

Since the first two terms are strictly positive by the first two Steps, we have $V_n''(p) > 0$. $\square$

B. Proof of Proposition 4. We work with likelihood ratios $Q(\ell) = e^\ell$, $Q_n \equiv q_n/(1-q_n)$ and $P_n \equiv p_n/(1-p_n)$. Fix an optimal policy, and in a further abuse of notation, define $\lambda_n^\theta(Q)$ as the transition chance (in θ) from belief Q to memory state n by the next distraction shock (as in (37) but replacing Q_n by Q); and define $w^\theta(Q)$ as the payoff starting from belief Q conditional on next shock a decision, in θ (given by (10), replacing Q_n by Q).

Step 1: *Distraction Shocks.* Let $G(Q)$ be the gain with *no* distraction shock at belief Q, namely $Q(V_n^H(Q) - \nu_n^H) + V_n^L(Q) - nu_n^L$. By (18) and (19), recalling $\nu_n^\theta = V_n^\theta(Q_n)$ from (12):

$$\begin{aligned} G(Q) &= \eta\left(Q(w^H(Q) - w^H(Q_n)) + (w^L(Q) - w^L(Q_n))\right) && (88) \\ &+ (1-\eta)\left(\sum_k Q(\lambda_k^H(Q) - \lambda_k^H(Q_n))\nu_k^H + (\lambda_k^L(Q) - \lambda_k^L(Q_n))\nu_k^L\right) && (89) \end{aligned}$$

Notice that $G(Q_n) = 0$, and so since G is convex in Q by Lemma 3), it suffices to prove that $G'(Q_n) = 0$, i.e. the gain reaches a minimum of 0 at $Q = Q_n$. For this, first consider the η coefficient in (88). The derivative in Q, evaluated at $Q = Q_n$, is $Q_n(w^H)'(Q_n) + (w^L)'(Q_n)$, which is zero by Step 1 in the Appendix B proof. Next consider (89). Using $\lambda_n^\theta(Q) = 1 - \sum_{k\neq n} \lambda_k^\theta(Q)$, rewrite it replacing each ν_k^θ with $\nu_k^\theta - \nu_n^\theta$ and summing only over states

$k \neq n$. Consider this sum for states $k \leq n-1$ (a symmetric argument applies for states $k \geq n+1$). First rewrite with cumulative transition chances and incremental payoff gaps:

$$\sum_{k\leq n-1} (\lambda_k^\theta(Q) - \lambda_k^\theta(Q_n))(\nu_k^\theta - \nu_n^\theta) = \sum_{k\leq n-1} \left(F^\theta \left(\log \frac{P_k}{Q} \right) - F^\theta \left(\log \frac{P_k}{Q_n} \right) \right) (\nu_k^\theta - \nu_{k+1}^\theta)$$

Now take Q times this expression in $\theta = H$, plus this expression in $\theta = L$ (this is the coefficient in (89) for memory states $k \leq n-1$). The derivative in Q, at $Q = Q_n$, is:

$$\sum_{k\leq n-1} \left[Q_n \frac{\partial}{\partial Q_n} \left(F^H \left(\log \frac{P_k}{Q_n} \right) \right) (\nu_k^H - \nu_{k+1}^H) + \frac{\partial}{\partial Q_n} \left(F^L \left(\log \frac{P_k}{Q_n} \right) \right) (\nu_k^L - \nu_{k+1}^L) \right] \quad (90)$$

But from (5), recalling that here $Q \geq P_k$ by construction, we have $F^H \left(\log \frac{P_k}{Q} \right) = \frac{\xi-1}{2\xi} \left(\frac{P_k}{Q_n} \right)^{\frac{\xi+1}{2}}$, and in state $\theta = L$ just swap $\xi+1$ and $\xi-1$. Using this, evaluate the derivative in (90) as follows, which vanishes (as desired) by the indifference FOC in Proposition 1:

$$\frac{\xi-1}{2\xi} \frac{\xi+1}{2Q_n} \left(\frac{P_k}{Q_n} \right)^{\frac{\xi-1}{2}} \sum_k \left[P_k(\nu_{k+1}^H - \nu_k^H) - (\nu_k^L - \nu_{k+1}^L) \right] \quad (91)$$

Step 2: *Decision Shocks.* We prove that if $Q < \pi/\Pi$, the gain to learning is positive $\forall Q \geq P_1$, while if $Q < P_1$ it is increasing in Q and negative at $Q = 0$. Converting (18) and (19) to likelihood ratios, $1+Q$ times the gain to learning is:

$$Q \left(\eta w^H(Q) + (1-\eta) \sum_{n=1}^{N} \lambda_n^H(Q) \nu_n^H \right) + \left(\eta w^L(Q) + (1-\eta) \sum_{n=1}^{N} \lambda^L(Q) \nu_n^L \right) - \pi \quad (92)$$

This is negative as $Q \downarrow 0$: For then by (37), Dug has zero chance of leaving state 1 by the next shock, so (92) reduces using (96) to $\eta\pi + (1-\eta)\nu_1^L - \pi$. This is negative if $\nu_1^L < \pi$, which holds since combining (12) and (20) yields $\nu_1^L < \eta w_1^L + (1-\eta)\nu_1^L \Rightarrow \nu_1^L < w_1^L$, while Corollary 1 implies that P_1 hence Q_1 is strictly positive, so by (10), $w_1^L < \pi$.

Next, decompose (92) as follows, letting $1 \leq k \leq N$ be the index with $P_{k-1} \leq Q < P_k$:

$$\eta \left(Q w^H(Q) + w^L(Q) \right) + (1-\eta) \left(Q \nu_k^H + \nu_k^L \right) - \pi \quad (93)$$

$$-(1-\eta) \sum_{n=1}^{k-1} (Q \lambda_n^H(Q)(\nu_k^H - \nu_n^H) - \lambda_n^L(Q)(\nu_n^L - \nu_k^L)) \quad (94)$$

$$+(1-\eta) \sum_{n=k+1}^{N} (Q \lambda_n^H(Q)(\nu_n^H - \nu_k^H) - \lambda_n^L(Q)(\nu_k^L - \nu_n^L)) \quad (95)$$

It remains to prove that this is increasing in $Q < P_1$ $(k = 1)$ and positive when $Q \geq P_1 (k \geq 2)$. First consider (94), if $k \geq 2$ (otherwise it vanishes). Since $Q \geq P_{k-1} \geq P_n$ for all terms in this sum, we have from (37) that $\lambda_n^H(Q) = \frac{\xi-1}{2}\left(\frac{P_n}{Q}\right)^{\frac{1}{2}(\xi+1)}\left(1-\gamma_{n-1}^{\frac{1}{2}(\xi+1)}\right)$, and just swap $\xi+1$ with $\xi-1$ to obtain $\lambda_n^L(Q)$. With this, using β from (40), (94) becomes $(1-\eta)$ times

$$\frac{\xi+1}{2\xi}\sum_{n=1}^{k-1}\left(\frac{P_n}{Q}\right)^{\frac{1}{2}(\xi-1)}\left(1-\gamma_{n-1}^{\frac{1}{2}(\xi-1)}\right)[(\nu_n^L-\nu_k^L)-\beta(\gamma_{n-1})P_n(\nu_k^H-\nu_n^H)]$$

This is positive, by $\beta(\gamma_{n-1}) < 1$ along with the Proposition 1 FOC $\frac{\nu_{n+1}^H-\nu_n^H}{\nu_n^L-\nu_{n+1}^L} = \frac{1}{P_n}$: Thus $P_n\frac{\nu_k^H-\nu_n^H}{\nu_n^L-\nu_k^L}$ is a weighted average of terms $\frac{P_n}{P_n}, \frac{P_n}{P_{n+1}}, ..., \frac{P_n}{P_{k-1}}$ which are at most 1 by $k-1 \geq n$. Similarly rewrite (95) as $1-\eta$ times the following, using transition chances for $Q \leq P_k \leq P_{n-1}$:

$$\frac{\xi+1}{2\xi}\sum_{n=k+1}^{N}\left(\frac{Q}{P_{n-1}}\right)^{\frac{1}{2}(\xi+1)}\left(1-\gamma_{n-1}^{\frac{1}{2}(\xi-1)}\right)[P_{n-1}(\nu_n^H-\nu_k^H)-\beta(\gamma_{n-1})(\nu_k^L-\nu_n^L)]$$

This is positive and increasing in Q, since each square bracketed term is positive again using $\beta \leq 1$ and the Proposition 1 implication that $P_{n-1}\frac{\nu_n^H-\nu_k^H}{\nu_k^L-\nu_n^L} \geq 1$.

Finally we prove that (93) is increasing in Q, and positive at $Q \geq P_1$. Clearly the coefficient on $1-\eta$ in (93) is increasing in Q, and to see that the coefficient on η is too, use (10) to rewrite it $(Qw^H(Q) + w^L(Q))$ as

$$Q\Pi\frac{\xi+1}{2\xi}\left(\frac{\Pi Q}{\pi}\right)^{\frac{1}{2}(\xi-1)}+\pi\left(1-\frac{\xi-1}{2\xi}\left(\frac{\Pi Q}{\pi}\right)^{\frac{1}{2}(\xi+1)}\right)=\pi\left(1+\frac{1}{\xi}\left(\frac{\Pi Q}{\pi}\right)^{\frac{1}{2}(\xi+1)}\right) \tag{96}$$

To complete the proof, we prove that (93) is positive at $Q = P_1$ $(k = 2)$. Since the coefficient on η exceeds π by (96)), it suffices to show that the one on $1-\eta$ does too, i.e. $P_1\nu_2^H + \nu_2^L \geq \pi$. For this, it suffices to prove that $Q_1\nu_1^H + \nu_1^L \geq \pi$, since by Proposition 1,

$$P_1\frac{\nu_2^H-\nu_1^H}{\nu_1^L-\nu_2^L} = 1 \Rightarrow P_1\nu_2^H+\nu_2^L = P_1\nu_1^H+\nu_1^L \geq Q_1\nu_1^H+\nu_1^L \tag{97}$$

Now rearrange (12) to obtain $\eta(\nu_1^\theta - w_1^\theta) = (1-\eta)\sum_{n\geq 2}\lambda_{1,n}^\theta(\nu_n^\theta-\nu_1^\theta)$. Taking ratios,

$$\frac{\nu_1^H-w_1^H}{w_1^L-\nu_1^L} = \frac{\sum_{n\geq 2}\lambda_{1,n}^H(\nu_n^H-\nu_1^H)}{\sum_{n\geq 2}\lambda_{1,n}^L(\nu_1^L-\nu_n^L)} > \frac{1}{Q_1}, \text{ since } \frac{\lambda_{1,n}^H(\nu_n^H-\nu_1^H)}{\lambda_{1,n}^L(\nu_1^L-\nu_n^L)} \equiv \frac{1}{\beta(\gamma_{n-1})}\frac{P_{n-1}}{Q_1}\frac{\nu_n^H-\nu_1^H}{\nu_1^L-\nu_n^L}$$

(which exceeds $\frac{1}{Q_1}$ again using $\beta < 1$ along with the FOC's in Proposition 1). So $Q_1\nu_1^H + \nu_1^L$ exceeds $Q_1w_1^H + w_1^L$, which by (96) at $Q = Q_1$ exceeds π. □.

F Proofs for Value Comparative Statics in Section 7

A. Proof of Proposition 5. To prove that Dug's value falls in both σ and δ, it suffices to prove that it falls in both $\eta \equiv \frac{\delta}{\alpha+\delta}$ and in $\xi \equiv \sqrt{1+2(\alpha+\delta)\sigma^2}$. For η, consider the value V^θ from (11). The terms w_n^θ from (10) do not depend on η, so we focus on the changes via ρ^θ. Recall that by (8), ρ^θ is the steady-state distribution of a perturbed Markov process where Dug is constrained, in every memory state, to jump to initial state n_0 with chance η. So lowering η relaxes this constraint, and thus Dug's optimized value falls in η. For ξ, differentiate (5) to see that F^H rises in ξ, while F^L falls in ξ. Thus raising ξ has two implications for any fixed policy: First, the expected values w_n^H and w_n^L from (10) both fall, (reducing the value). Second, the cdf $\sum_{k=1}^K \lambda_{n,k}^\theta$ rises when $\theta = H$, and falls when $\theta = L$. In state H, this induces a first-order stochastic decrease in $\lambda_{n,k}^H$, thus reducing the expectation of increasing (by Lemma 2) function ν_n^H, and hence reducing V^H. And similarly in state L, where a first order increase lowers the expectation of decreasing function ν_n^L.

B. Proof of Proposition 6. For the limits as $\alpha \downarrow 0$ and $N \uparrow \infty$: That the limits are the same follows from (21), which at $\alpha = 0$ reduces to the standard HJB equation, reflecting rational ($N = \infty$) learning. The $\alpha = 0$ limit value was explained in the text.

For the limits $\sigma \to 0$, $\delta \to \infty$, $\sigma \to \infty$, and $\alpha \to \infty$: If $\mathcal{V}^*(1)$ and $\mathcal{V}^*(\infty)$ share a limit, then this must be the limit for all N, since $\mathcal{V}^*(N)$ is increasing in N. This logic yields the result for $\alpha + \delta \to 0$, $\sigma \to 0$, $\delta \to \infty$, and $\sigma \to \infty$. Now consider the $\alpha \to \infty$ limit, so $\eta \to 1$ and $\xi \to \infty$. Then $\rho_{n_0}^\theta \to 1$, and so $Q_{n_0} \to \mu$, thus $F^\theta(\log(\pi/(\Pi Q_{n_0}))) \to 0$, and so $V^*(N) \to \mu\Pi$ by (11).

Finally, for the limit $\delta \downarrow 0$: Since $\mathcal{V}^*$ increases in N, it suffices to prove that Dug earns $\mathcal{V}^{FI}$ with two memory states. From Section 5, beliefs are given by $Q_1 = P_1\underline{x}$ and (16), where $\underline{x}$ is the root of equation (38). As δ and hence $\eta \to 0$, $\underline{x} \to 0$ by inspection of (38). But then for *any* $P_1 > 0$, $Q_1 = P_1\underline{x} \to 0$. And $Q_2 \to \infty$, since the RHS of (16) tends to 0 with $\underline{x}$, thus the LHS must too, requiring $Q_2 \to \infty$. And so taking limits in (86) as $Q_1 \to 0$ and $Q_2 \to \infty$ (recalling $\xi > 1$), we obtain Dug's limit value:

$$\begin{aligned}
\lim_{\delta\to 0} \mathcal{V}^\star(2) &= \pi \lim_{Q_2\uparrow\infty, Q_1\downarrow 0} \left(\frac{Q_2-\mu}{Q_2-Q_1} \left(\frac{1}{\xi} \left(\frac{\Pi Q_1}{\pi} \right)^{\frac{\xi+1}{2}} + 1 \right) + \frac{\mu - Q_1}{Q_2-Q_1} \left(\frac{\Pi Q_2}{\pi} + \frac{1}{\xi} \left(\frac{\pi}{\Pi Q_2} \right)^{\frac{\xi-1}{2}} \right) \right) \\
&= \pi \lim_{Q_2\uparrow\infty} \left(\frac{Q_2-\mu}{Q_2} + \frac{\mu}{Q_2}\frac{\Pi Q_2}{\pi} \right) = \pi + \mu\Pi \qquad \square
\end{aligned}$$

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